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FLUTTER OF A UNIFORM WING WITH AN ARBITRARILY PLACED MASS ACCORDING TO A DIFFERENTIAL-EQUATION ANALYSIS AND A COMPARISON WITH EXPERIMENT

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Page 9 in the subject paper is incorrect. Consequently, page 9 in the copy should be replaced by the attached correct copy of page 9. This page has been backed up by page 10 to facilitate replacement in the proper place.

Page 25, equation (A16) should have the term $- f^{n-1}(0)$ inserted after the term $- sf^{n-2}(0)$.

Solving equations (3) and (4) for $\bar{y}(s)$ and $\bar{\theta}(s)$ gives the Laplace transform of $y(x)$ and $\theta(x)$, respectively, as

$$\bar{y}(s) = \frac{\left(s^3 + s\delta\right)Y_2 + \left(s^2 + \delta\right)Y_3 + \beta\theta_1 - \beta e^{-sl_1}\left[\theta'\left(l_1 - 0\right) - \theta'\left(l_1 + 0\right)\right]}{q(s)}$$

$$- \frac{\left(s^2 + \delta\right)e^{-sl_1}\left[y'''\left(l_1 - 0\right) - y'''\left(l_1 + 0\right)\right]}{q(s)} \tag{5}$$

and

$$\bar{\theta}(s) = \frac{s^4\theta_1 - \gamma s Y_2 - \gamma Y_3 - \theta_1\alpha + \gamma e^{-sl_1}\left[y'''\left(l_1 - 0\right) - y'''\left(l_1 + 0\right)\right]}{q(s)}$$

$$+ \frac{\left(\alpha - s^4\right)e^{-sl_1}\left[\theta'\left(l_1 - 0\right) - \theta'\left(l_1 + 0\right)\right]}{q(s)} \tag{6}$$

where

$$q(s) = s^6 + \delta s^4 - \alpha s^2 + \gamma\beta - \alpha\delta$$

Goland and Luke (reference 4) showed that $y(x)$ and $\theta(x)$ could be written as a converging series by expanding the transforms (5) and (6) in terms of symmetric polynomials of the squares of the roots of $q(s)$ and applying the inverse transform. A discussion of this expansion is given in section 4 of appendix A where it is shown that $1/q(s)$ can be written as

$$\frac{1}{q(s)} = \frac{1}{s^6}\sum_{n=0}^{\infty}\frac{T_n}{s^{2n}} \tag{7}$$

where

$$T_0 = 1$$

$$T_1 = -\delta$$

$$T_2 = \delta^2 + \alpha$$

$$T_3 = -\delta^3 - \alpha\delta - \beta\gamma$$

$$\cdot \cdot \cdot \cdot \cdot$$

For $n \geq 3$,

$$T_n = -\delta T_{n-1} + \alpha T_{n-2} + (\alpha\delta - \beta\gamma)T_{n-3} \tag{8}$$

When the series expansion of $1/q(s)$, equation (7), is substituted into equations (5) and (6) the transforms $\bar{y}(s)$ and $\bar{\theta}(s)$ become sums of infinite series with terms of two distinct types; that is, terms of types

$$\frac{A}{s^m}$$

and

$$\frac{Be^{-sx_o}}{s^m}$$

where m is a positive integer.

The inverse Laplace transform of $\frac{A}{s^m}$ (see pair no. 3, p. 295, of reference 9) for $x > 0$ is

$$L^{-1}\left\{\frac{A}{s^m}\right\} = \frac{Ax^{m-1}}{(m-1)!} \tag{9}$$

and the inverse Laplace transform of $\frac{Be^{-sx_o}}{s^m}$ (see pair no. 63, p. 298, of reference 9) for $x > x_o \geq 0$ is

$$L^{-1}\left\{\frac{Be^{-sx_o}}{s^m}\right\} = \frac{B(x - x_o)^{m-1}}{(m-1)!} \tag{10}$$

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SUMMARY

A method is presented for the calculation of the flutter speed of a uniform wing carrying an arbitrarily placed concentrated mass. The method, an extension of recently published work by Goland and Luke, involves the solution of the differential equations of motion of the wing at flutter speed and therefore does not require the assumption of specific normal modes of vibration. The order of the flutter determinant to be solved by this method depends upon the order of the system of differential equations and not upon the number of modes of vibration involved.

The differential equations are solved by operational methods and a brief discussion of operational methods as applied to boundary-value problems is included in one of two appendixes. A comparison is made with experiment for a wing with a large eccentrically mounted weight and good agreement is obtained. Sample calculations are presented to illustrate the method; and curves of amplitudes of displacement, torque, and shear for a particular case are compared with corresponding curves computed from the first uncoupled normal modes.

For convenience, the method employs two-dimensional air forces and could be extended to apply to uniform wings with any number of arbitrarily placed concentrated weights, one of which might be considered as a fuselage. The location of such masses as engines, fuel tanks, and landing-gear installations might be used to advantage in increasing the flutter speed of a given wing.

INTRODUCTION

The common procedures in flutter analysis of an airplane wing involve many simplifying assumptions. In particular the degrees of freedom of the wing are usually determined by choosing the first few normal modes of the structure, and the wing motion at flutter is then described in terms of these chosen modes. This approach of employing prescribed modes is often adapted to the Rayleigh type analysis of vibration and may be referred to as "Rayleigh type analysis." In specific calculations with this method the amount of work required is proportional to the number of normal modes involved. In particular, the order of the flutter determinant that must be solved depends directly upon the number of modes involved. For simple wings, without concentrated masses, the Rayleigh type analysis usually yields satisfactory results with not more than two or three normal modes. However, if the wing carries concentrated masses, such as engine, fuel tank, or landing-gear installations, so many normal modes may be required to obtain satisfactory results that the Rayleigh method may not be the most feasible method.

In cases where many degrees of freedom are involved the most logical procedure would be to treat the system of differential equations of motion of the wing rather than to choose specific modes. This method is in general very difficult and tedious to carry through, although it has the advantage that the order of the flutter determinant that must be solved depends only upon the order of the system of differential equations and not upon the number of modes of vibration involved.

As early as 1929 Küssner (reference 1) used the differential equation approach to formulate the problem in the form of an integro-differential equation for a wing of general plan form. Küssner set up some particular examples and suggested a method of solution by a process of iteration. This method was not followed up until during the war when some related work was undertaken in Germany but not finished. Wielandt (reference 2) has recently made contributions to the treatment of nonself-adjoint differential equations by iterative processes. In the light of these contributions perhaps the problem of flutter analysis as proposed by Küssner warrants further investigation.

Recently, Goland (reference 3) applied the differential-equation method to a uniform cantilever wing and was able to carry out the solution of the flutter problem by straightforward methods. In reference 4 Goland and Luke extended the solution of the problem of the uniform wing to include a uniform wing carrying a fuselage at the semispan and concentrated weights at the tips. Goland and Luke made use of the Laplace transform to solve the differential equations by operational methods for both the symmetric and antisymmetric types of flutter. In both references 3 and 4, the objective was to compare flutter speeds and certain flutter parameters for specific uniform wings calculated by the differential-equations method with the same quantities calculated by the Rayleigh method when only the fundamental bending and torsion modes were used in the calculations. Fairly close agreement between results calculated by the two methods were obtained in both references 3 and 4. No comparison with experiment, however, was made in either case.

The results of a systematic series of flutter tests made to determine the effect of concentrated weights and concentrated weight positions on the flutter speed of a uniform cantilever wing are reported in reference 5. After these experiments were finished, an attempt was made to compare the results with a theoretical analysis by the Rayleigh method. In cases where the mass of the weight was of the same order as that of the wing and placed so that the distance between its center of gravity and the elastic axis of the wing was a considerable fraction of the wing chord, however, several normal modes would have to be employed and there was no way of knowing in advance just what number should be used. Because of this difficulty and because the wing was a uniform wing, the most extreme case was chosen from reference 5 and investigated by the differential-equations method by following an extended procedure of Goland and Luke. The purpose of this paper is to report the results of this investigation.

The paper consists of the main text and two appendixes. In the main text the differential-equation method is set up for any uniform cantilever wing with an arbitrarily placed concentrated weight and the solution, based on an extension of the method used by Goland and Luke, is developed. Application is then made to a particular wing-weight system used in reference 5, and comparison with experimental results is given. The mass of the weight (weight labeled 7a in reference 5) was about 92 percent of the mass of the wing and at each spanwise weight position the weight was placed so that its center of gravity was about 0.41 chord forward of the elastic axis of the wing. (It may be mentioned for the sake of comparison that in the numerical example treated in reference 4, the mass of the weight was only 39 percent of the mass of the wing and placed 0.1 chord behind the elastic axis of the wing.) The geometric aspect ratio of the wing was 6, which was considered large enough to warrant the use of two-dimensional air forces without aspect-ratio corrections for oscillating instability (not necessarily so for the divergent type of instability (see reference 6)). One other simplification was the omission of terms due to structural damping. The computed results agree remarkably well with experimental results, particularly in regard to trends.

In appendix A the method used by Goland and Luke, which includes the derivation of the differential equations, for a wing carrying a tip weight is outlined and extended to a wing carrying an arbitrarily placed weight. A somewhat general but brief discussion of operational methods of solving boundary-value problems is included and illustrated with a simple example for readers who might be interested but are not familiar with the operational approach.

In appendix B the derivation of the flutter determinant is completed and a method of solving the determinant is illustrated by a detailed calculation of the flutter speed for the wing and one weight position of the wing-weight combination discussed in the test. As a final topic in this appendix the solution obtained for the flutter determinant is used with the solutions of the differential equations to calculate the amplitudes and phase angles of the deflection curves of the wing-weight system at flutter speed.

SYMBOLS

(The symbols are given in terms of a consistent set of units that are convenient for the computations in this paper. They can be converted to any desired set of units by proper attention to the dimensions involved.)

a — nondimensional distance of elastic axis from midchord measured in half-chords, positive for positions of elastic axis behind midchord

b — wing half-chord, feet

e_1 — chordwise distance of wing center of gravity from elastic axis, positive for center of gravity behind elastic axis, feet

e_2 — chordwise distance of weight center of gravity from elastic axis, positive for center of gravity behind elastic axis, feet

g — gravitational constant, feet per second per second

I — mass moment of inertia of uniform wing per unit of spanwise length, referred to wing elastic axis, pound-second^2 $\left(MK_1^2\right)$

I_w — mass moment of inertia of weight referred to wing elastic axis, foot-pound-second^2

K_1 — radius of gyration of wing sections about wing elastic axis, feet

K_2 — radius of gyration of weight about elastic axis, feet

k — reduced-frequency parameter $\left(\frac{b\omega}{v}\right)$

L — aerodynamic lift force per unit of spanwise length

$$L_y + iL_y' = \pi\rho b^2 L_h$$

$$L_\theta + iL_\theta' = \pi\rho b^3\left[L_\alpha - L_h\left(\frac{1}{2} + a\right)\right]$$

l — semispan of wing, feet

l_1 — location of weight measured from wing root, feet

$L_h, L_\alpha, M_h, M_\alpha$	aerodynamic coefficients as tabulated in reference 7
M	aerodynamic moment per unit of spanwise length taken about elastic axis

$$M_y + iM_y' = \pi\rho b^3\left[M_h - L_h\left(\frac{1}{2} + a\right)\right]$$

$$M_\theta + iM_\theta' = \pi\rho b^4\left[M_\alpha - L_\alpha\left(\frac{1}{2} + a\right) - M_h\left(\frac{1}{2} + a\right) + L_h\left(\frac{1}{2} + a\right)^2\right]$$

W	weight of wing model, pounds
m	mass of wing per unit length
W_w	weight of concentrated weight, pounds
N	transverse shear force in wing at station x
T	torsional moment in wing at station x
R_1, R_2, R_3	roots of cubic equation
s	operator used in Laplace transformation
t	time coordinate
T_n	sum of all symmetric polynomial functions in R_1, R_2, R_3 which are of degree n
v_o	experimental flutter speed for wing without weight, feet per second
v	flutter speed, feet per second
$\frac{v}{b\omega}$	reduced flutter speed
x	spanwise coordinate measured from wing root
$Y(x,t)$	general mode shape function in bending
$y(x)$	mode shape function in bending after assumption of harmonic motion $\left(y_1(x) + iy_2(x)\right)$
EI_b	flexural rigidity of uniform wing, pound-feet^2
GJ	torsional rigidity of uniform wing, pound-feet^2

$$\alpha = \frac{\omega^2}{EI_b}\left(m + L_y + iL_y{}'\right)$$

$$\beta = \frac{\omega^2}{EI_b}\left(me_1 + L_\theta + iL_\theta{}'\right)$$

$$\gamma = \frac{\omega^2}{GJ}\left(me_1 + M_y + iM_y{}'\right)$$

$$\delta = \frac{\omega^2}{GJ}\left(I + M_\theta + iM_\theta{}'\right)$$

κ	mass ratio $\left(\frac{\pi\rho b^2}{m}\right)$
ρ	air density, slugs per cubic foot
Δ	complex value of determinant
Δ_e	value of Δ when real and imaginary parts are equal
$\Theta(x,t)$	general mode shape function in torsion
$\theta(x)$	mode shape function in torsion after assumption of harmonic motion $\left(\theta_2(x) + i\theta_3(x)\right)$
ω	circular frequency at flutter, radians per second
f	frequency, cycles per second $\left(\frac{\omega}{2\pi}\right)$

ANALYSIS

As mentioned in the introduction the differential equations that govern the motion of a uniform wing at flutter speed, as derived by Goland in reference 3, and a method of solving the equations for a uniform cantilever wing carrying an arbitrarily placed weight, based on a method developed by Goland and Luke in reference 4, are discussed in appendix A. This section, therefore, is devoted to a brief discussion of the differential equations of motion of the wing, the boundary conditions, solution of the boundary-value problem by means of the Laplace transform, and the solution of the flutter determinant.

The differential equations and boundary conditions that govern the motion, at flutter speed, of a cantilever wing of length l with a concentrated weight placed l_1 units along the span from the root section and e_2 units forward of the elastic axis of the wing, as derived in appendix A, are

$$y^{iv}(x) - \alpha y(x) - \beta\theta(x) = 0 \tag{1}$$

$$\theta''(x) + \gamma y(x) + \delta\theta(x) = 0 \tag{2}$$

(a) $y(0) = y'(0) = \theta(0) = 0$

(b) $EI_b y''(l) = EI_b y'''(l) = GJ\theta'(l) = 0$

(c) $EI_b\left[y'''(l_1 - 0) - y'''\left(l_1 + 0\right)\right] = -\frac{W_w}{g}\omega^2\left[y(l_1) + e_2\theta\left(l_1\right)\right]$

(d) $GJ\left[\theta'\left(l_1 - 0\right) - \theta'\left(l_1 + 0\right)\right] = \frac{W_w}{g}\omega^2\left[e_2 y\left(l_1\right) + K_2{}^2\theta\left(l_1\right)\right]$

where

$$\alpha = \frac{\omega^2}{EI_b}\left(m + L_y + iL_y{}'\right)$$

$$\beta = \frac{\omega^2}{EI_b}\left(me_1 + L_\theta + iL_\theta{}'\right)$$

$$\gamma = \frac{\omega^2}{GJ}\left(me_1 + M_y + iM_y{}'\right)$$

$$\delta = \frac{\omega^2}{GJ}\left(I + M_\theta + iM_\theta{}'\right)$$

and where $y(x)$ is the displacement of a chordwise element of the elastic axis of the wing at span position x due to bending; $\theta(x)$ is the corresponding displacement due to torsion; primes associated with y and θ indicate differentiation with respect to x; EI_b is the flexural rigidity of the wing; GJ is the torsional rigidity of the wing; $\frac{W_w}{g}$ is mass of the weight; m is mass per unit length of wing; and ω is the circular frequency of bending and torsion at flutter. In condition (c) the notation $y'''(l_1 - 0)$ indicates that $y'''(x)$ is to have the value that it approaches as $x \rightarrow l_1$ from the inboard side of the weight and $y'''(l_1 + 0)$ indicates that $y'''(x)$ is to have the value that it approaches as $x \rightarrow l_1$ from the outboard side of the weight. Similar meanings are given to $\theta'(l_1 - 0)$ and $\theta'(l_1 + 0)$.

The quantities $L_y + iL_y'$, $L_\theta + iL_\theta'$, $M_y + iM_y'$, and $M_\theta + iM_\theta'$ can be written in terms of tabulated quantities as follows:

$$L_y + iL_y' = \pi\rho b^2 L_h$$

$$L_\theta + iL_\theta' = \pi\rho b^3\left[L_\alpha - L_h\left(\frac{1}{2} + a\right)\right]$$

$$M_y + iM_y' = \pi\rho b^3\left[M_h - L_h\left(\frac{1}{2} + a\right)\right]$$

$$M_\theta + iM_\theta' = \pi\rho b^4\left[M_\alpha - L_\alpha\left(\frac{1}{2} + a\right) - M_h\left(\frac{1}{2} + a\right) + L_h\left(\frac{1}{2} + a\right)^2\right]$$

In reference 7 the values of L_h, L_α, M_h, and M_α are expressed in terms of Theodorsen's F and G functions of reference 8 and tabulated for various values of the reduced speed $v/b\omega$.

The root conditions (a) and the boundary conditions (b), of the boundary-value problem, are the usual conditions that must be imposed upon the equations of a vibrating cantilever beam (or wing). Conditions (c) and (d) stipulate discontinuities of determinable magnitudes in transverse shearing force and torque, respectively.

Applying the Laplace transform (see appendix A)

$$\int_0^\infty e^{-sx}f(x)\ dx = \bar{f}(s)$$

to equations (1) and (2) and making use of conditions (a), (c), and (d) gives

$$s^4\bar{y}(s) - sY_2 - Y_3 + e^{-sl_1}\left[y'''(l_1 - 0) - y'''(l_1 + 0)\right] - \alpha\bar{y}(s) - \beta\bar{\theta}(s) = 0 \tag{3}$$

and

$$s^2\bar{\theta}(s) - \theta_1 + e^{-sl_1}\left[\theta'(l_1 - 0) - \theta'(l_1 + 0)\right] + \delta\bar{\theta}(s) + \gamma\bar{y}(s) = 0 \tag{4}$$

where

$$Y_2 = y''(0)$$

$$Y_3 = y'''(0)$$

$$\theta_1 = \theta'(0)$$

and

$$\theta(x) = \theta_1 \sum_{n=0}^{\infty} \frac{T_n x^{2n+1}}{(2n + 1)!} - \gamma Y_2 \sum_{n=0}^{\infty} \frac{T_n x^{2n+4}}{(2n + 4)!} - \gamma Y_3 \sum_{n=0}^{\infty} \frac{T_n x^{2n+5}}{(2n + 5)!}$$

$$- \theta_1 \alpha \sum_{n=0}^{\infty} \frac{T_n x^{2n+5}}{(2n + 5)!} + \left[\theta'(l_1 - 0) - \theta'(l_1 + 0)\right] \left[\alpha \sum_{n=0}^{\infty} \frac{T_n \left(x - l_1\right)^{2n+5}}{(2n + 5)!}\right.$$

$$\left. - \sum_{n=0}^{\infty} \frac{T_n \left(x - l_1\right)^{2n+1}}{(2n + 1)!}\right]$$

$$+ \gamma \left[y'''\left(l_1 - 0\right) - y'''\left(l_1 + 0\right)\right] \sum_{n=0}^{\infty} \frac{T_n \left(x - l_1\right)^{2n+5}}{(2n + 5)!} \qquad (12)$$

where in both equation (11) and equation (12) the terms involving $\left(x - l_1\right)$ are to be considered as zero when $x = l_1$.

Equations (11) and (12) are general expressions for the amplitudes or displacements of a point x of the elastic axis of a uniform wing vibrating in bending and torsion under the conditions of flutter with an arbitrarily placed concentrated weight. When the weight is concentrated at the wing tip the equations correspond to those obtained by Goland except for a difference in root conditions. When the weight is concentrated at the root (or if the mass of the weight is reduced to zero) the equations reduce to those for a uniform cantilever wing. These equations may appear rather formidable in their present form; however, only the first few terms of each summation seem necessary for most cases.

In the derivation of the flutter determinant in appendix B it is shown that since terms involving $(x - l_1)$ drop out of both equation (11) and equation (12) at $x = l_1$, the values of $y(l_1)$ and $\theta(l_1)$ can be obtained from the terms not involving $(x - l_1)$. Then, by making use of conditions (c) and (d) again, linear expression in Y_2, Y_3, and θ_1 can be substituted for the bracketed expressions

$$\left[y'''\left(l_1 - 0\right) - y'''\left(l_1 + 0\right)\right]$$

For $n \geq 3$,

$$T_n = -\delta T_{n-1} + \alpha T_{n-2} + (\alpha\delta - \beta\gamma) T_{n-3} \tag{8}$$

When the series expansion of $1/q(s)$, equation (7), is substituted into equations (5) and (6) the transforms $\bar{y}(s)$ and $\bar{\theta}(s)$ become sums of infinite series with terms of two distinct types; that is, terms of types

$$\frac{A}{s^m}$$

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where m is a positive integer.

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$$L^{-1}\left\{\frac{A}{s^m}\right\} = \frac{Ax^{m-1}}{(m - 1)!} \tag{9}$$

and the inverse Laplace transform of $\frac{Be^{-sx_o}}{s^m}$ (see pair no. 63, p. 298, of reference 9) for $x > x_o \geq 0$ is

$$L^{-1}\left\{\frac{Be^{-sx_o}}{s^m}\right\} = \frac{B(x - x_o)^{m-1}}{(m - 1)!} \tag{10}$$

When the expression for $1/q(s)$ from equation (7) is substituted into equations (5) and (6) and the inverse transforms is applied, the following series expressions of $y(x)$ and $\theta(x)$ can be obtained:

$$
\begin{aligned}
y(x) = {} & Y_2\left[\sum_{n=0}^{\infty}\frac{T_n x^{2n+2}}{(2n+2)!} + \delta\sum_{n=0}^{\infty}\frac{T_n x^{2n+4}}{(2n+4)!}\right] + Y_3\left[\delta\sum_{n=0}^{\infty}\frac{T_n x^{2n+5}}{(2n+5)!}\right. \\
& \left. + \sum_{n=0}^{\infty}\frac{T_n x^{2n+3}}{(2n+3)!}\right] + \theta_1\beta\sum_{n=0}^{\infty}\frac{T_n x^{2n+5}}{(2n+5)!} \\
& - \beta\left[\theta'(l_1 - 0) - \theta'(l_1 + 0)\right]\sum_{n=0}^{\infty}\frac{T_n (x - l_1)^{2n+5}}{(2n+5)!} \\
& - \left[y'''(l_1 - 0) - y'''(l_1 + 0)\right]\left[\delta\sum_{n=0}^{\infty}\frac{T_n (x - l_1)^{2n+5}}{(2n+5)!}\right. \\
& \left. + \sum_{n=0}^{\infty}\frac{T_n (x - l_1)^{2n+3}}{(2n+3)!}\right]
\end{aligned}
\tag{11}
$$

and

$$\theta(x) = \theta_1 \sum_{n=0}^{\infty} \frac{T_n x^{2n+1}}{(2n + 1)!} - \gamma Y_2 \sum_{n=0}^{\infty} \frac{T_n x^{2n+4}}{(2n + 4)!} - \gamma Y_3 \sum_{n=0}^{\infty} \frac{T_n x^{2n+5}}{(2n + 5)!}$$

$$- \theta_1 \alpha \sum_{n=0}^{\infty} \frac{T_n x^{2n+5}}{(2n + 5)!} + \left[\theta'(l_1 - 0) - \theta'(l_1 + 0)\right] \left[\alpha \sum_{n=0}^{\infty} \frac{T_n (x - l_1)^{2n+5}}{(2n + 5)!}\right.$$

$$\left. - \sum_{n=0}^{\infty} \frac{T_n (x - l_1)^{2n+1}}{(2n + 1)!}\right]$$

$$+ \gamma \left[y'''(l_1 - 0) - y'''(l_1 + 0)\right] \sum_{n=0}^{\infty} \frac{T_n (x - l_1)^{2n+5}}{(2n + 5)!} \qquad (12)$$

where in both equation (11) and equation (12) the terms involving $(x - l_1)$ are to be considered as zero when $x = l_1$.

Equations (11) and (12) are general expressions for the amplitudes or displacement of a point x of the elastic axis of a uniform wing vibrating in bending and torsion under the conditions of flutter with an arbitrarily placed concentrated weight. When the weight is concentrated at the wing tip the equations correspond to those obtained by Goland except for a difference in root conditions. When the weight is concentrated at the root (or if the mass of the weight is reduced to zero) the equations reduce to those for a uniform cantilever wing. These equations may appear rather formidable in their present form; however, only the first few terms of each summation seem necessary for most cases.

In the derivation of the flutter determinant in appendix B it is shown that since terms involving $(x - l_1)$ drop out of both equation (11) and equation (12) at $x = l_1$, the values of $y(l_1)$ and $\theta(l_1)$ can be obtained from the terms not involving $(x - l_1)$. Then, by making use of conditions (c) and (d) again, linear expression in Y_2, Y_3, and θ_1 can be substituted for the bracketed expressions

$$\left[y'''(l_1 - 0) - y'''(l_1 + 0)\right]$$

and

$$\left[\theta'(l_1 - 0) - \theta'(l_1 + 0)\right]$$

After the substitutions are made, equations (11) and (12) will contain only the three undetermined coefficients Y_2, Y_3, and θ_1 for any particular wing-weight system of the type under consideration. Observe that conditions (b) have not yet been used. If these conditions are now imposed upon the equations, there is obtained a system of three linear homogeneous equations in Y_2, Y_3, and θ_1 that may be written for reference as

$$A_iY_2 + B_iY_3 + C_i\theta_1 = 0 \tag{13}$$

where $i = 1, 2,$ and 3.

The condition that a system of equations such as equations (13) have solutions other than the trivial solution

$$Y_2 = Y_3 = \theta_1 = 0$$

is that the determinant of the coefficients A_i, B_i, and C_i vanish (reference 10). This corresponds to the border-line condition between damped (stable) and undamped (unstable) oscillations or to the point at which flutter occurs. It will be noted that the order of this determinant depends only on the order of the system of differential equations.

The actual coefficients corresponding to A_i, B_i, and C_i are complex functions of the frequency ω, the reduced flutter speed $v/b\omega$, and certain determinable characteristics of the wing-weight system. The true flutter speed is easily calculated when corresponding values of ω and $v/b\omega$ are known. These quantities may therefore be considered as (the only) variable parameters in the determinant of coefficients and the problem of finding the true flutter speed is reduced to that of finding corresponding values of these parameters that cause the determinant, hereinafter called the flutter determinant, to vanish. If v is set equal to zero the air forces drop out and the resulting determinant gives the coupled modes of vibration of the wing in still air. On the other hand, if ω is set equal to zero the nonoscillatory or divergence condition is obtained.

Several ways of solving the flutter determinant are mentioned in reference 6. Although more informative methods exist, a graphical method was adopted for the present work. For example, a value is assigned to one parameter, preferably $v/b\omega$; the flutter determinant is then evaluated for this value of $v/b\omega$ and several values of the other parameter ω. The values of the flutter determinant obtained in this manner are complex numbers and if the real and imaginary parts of a sufficient number of determinant values are separately plotted against ω, the point or points where the real and imaginary parts are equal are obtained. If this process for other values of $v/b\omega$ is repeated, a locus of determinant values with equal real and imaginary parts can be plotted against both $v/b\omega$ and ω. When enough points are determined these plots give the values of $v/b\omega$ and ω that cause the determinant to vanish.

An illustration of the process of solving the flutter determinant as described in the preceding paragraph is given in appendix B which contains the complete solution of the determinant for one weight position of the particular wing-weight system described in the section entitled "Application to a Specific Wing-Weight System." In general, when solving the flutter determinant by the preceding method, if the assumed values of $v/b\omega$ and ω are in the neighborhood of their true values, only a few points need be computed to obtain a solution. In the absence of experimental values of these parameters and in view of the work involved in determining other parameters that depend on $v/b\omega$, it will be found advisable to use simplified methods to obtain approximate values with which to start the solution.

APPLICATION TO A SPECIFIC WING-WEIGHT SYSTEM

Attention is now turned to the application of the boundary-value problem discussed in the foregoing section to a specific problem. The wing-weight system that has been analyzed consists of a particular uniform cantilever wing and weight combination described in reference 5. The weight was considered as concentrated at different specified span positions but always at about 0.41 chord forward of the elastic axis of the wing. This weight was selected because of its high mass compared to that of the wing and because of the large eccentricity due to the distance between its center of gravity and the elastic axis of the wing. Furthermore, by using only the fundamental modes, first bending and first torsion, the Rayleigh type analysis had failed to give any reasonable results for this particular wing-weight combination. Pertinent data, based on measured characteristics of the wing as taken from reference 5, with the units in feet and pounds are

Chord, feet	2/3
Length, feet	4
Aspect ratio (geometric)	6
Taper ratio	1
Airfoil section	NACA 16-010
W, pounds	3.48
I, pound-second^2	0.00080
EI_b, pound-feet^2	977.08
GJ, pound-feet^2	480.56
$1/\kappa$ (standard air, no weight)	32.6
e_1, feet	0.013

and, based on measured characteristics of the weight, are

W_w, pounds	3.182
e_2, feet	-0.2728
I_w, foot-pound-second^2	0.013625

Calculation of the flutter parameters have been made for the wing without the weight and for the wing with the weight at seven different positions. The calculated results are compared with experimental results in figure 1 and in the following table:

l_1 (in.)	e_2 (ft)	Calculated			Experimental		
		f (cps)	$v/b\omega$	v (fps)	f (cps)	$v/b\omega$	v (fps)
0	-----	25.27	6.29	333	22.1	7.22	334
11	-0.2728	19.23	8.23	331	17.4	8.88	324
17	-.2728	28.04	6.93	407	[a]26.8	6.81	382
30	-.2728	30.68	8.18	526	(b)	(b)	(b)
45	-.2728	25.67	7.45	401	(b)	(b)	(b)
46	-.2728	24.87	7.06	368	21.8	8.06	368
48	-.2728	23.60	6.07	300	21.4	7.14	320

[a]It is found in reference 5 that good flutter records for this wing-weight system were obtained for several spanwise weight positions between the root section and a point 17 inches from the root section; but with the weight at 17 inches from the root section the wing appeared to diverge. However, the oscillograph records for this case showed two possible flutter points, one corresponding to a frequency of 16.3 cps and another corresponding to a frequency of 26.8 cps (only the first of these is recorded in reference 5). When the weight was moved farther outward from this point, definite divergence was noted until the weight was at a point 46 inches from the root section. At this point and from this point to the tip good flutter records were obtained.

[b]Divergence.

It will be noted in the table that all the calculated flutter speeds are within 7 percent of the experimental values and the calculated frequencies and reduced speeds are within 15 percent of the experimental values. The calculated flutter speeds are generally slightly higher than the experimental values for $l_1 \leq 17$ and slightly lower for $l_1 \geq 46$. There is no such consistent trend in the other parameters.

In figure 1 the ratio of both calculated and experimental flutter speeds for the wing with a weight to the flutter speed of the wing without a weight are plotted against span position of the weight. The important thing to note in examining figure 1 is that the shape of the theoretical curve follows the shape of the experimental curve very closely in the regions where experimental flutter was obtained. The horizontal dashed line in figure 1 represents the divergence speed for the wing as computed by the method of reference 11. Although the correct divergence speed for different weight positions would probably vary, being somewhat lower with the weight at the tip than at the root, owing to the effect of the presence of the weight on aerodynamic forces, the agreement of the approximate value with experimental values is satisfactory.

General expressions for the deflection curves are derived in appendix B from which amplitudes and phase angles for curves of deflection, slope, moment, and shear in bending and amplitudes and phase angles for curves of angular deflection and torque in torsion can be computed. The phase angles and amplitudes for the deflection and shear curves in bending (fig. 2) and the phase angles and amplitudes for the angular displacement and torque in torsion (fig. 3) have been computed with reference to a unit tip deflection for the weight position $l_1 = 17$ inches. In figure 4 the amplitudes in deflection and shear in bending from figure 2 are compared with the deflection and shear curves due to the fundamental uncoupled bending mode of the wing, and in figure 5 the amplitudes in angular deflection and torque in torsion from figure 3 are compared with the angular deflection and torque curves due to the fundamental uncoupled mode in torsion. There is a notable difference in the shape of the amplitude curves computed by the present method and those computed from the first normal modes. This discrepancy indicates that several modes would have to be employed to obtain satisfactory results by the Rayleigh type analysis.

CONCLUDING REMARKS

The method discussed in this paper is not limited to a uniform cantilever wing with a single weight. By proper attention to the boundary conditions the theory can quite easily be extended to apply to a uniform wing carrying any number of arbitrarily placed weights, one of which might be considered as a fuselage and made to yield the so-called symmetric and antisymmetric types of flutter. Furthermore, for convenience of application, theoretical values of two-dimensional air forces have been used. However, since the method does not depend on the particular form of air forces involved, any known or available aerodynamic data could be used. In any event, the method is tedious and would, therefore, not be recommended over the Rayleigh type analysis when it might be known that only the first few normal modes of the structure are sufficient to give satisfactory results.

For wings that are not uniform the differential equations for flutter conditions reduce to ordinary differential equations with variable coefficients. In this case the solution would, in general, be much more difficult to obtain. For general cases there would be no advantage in the operational method of solution although an iterative process probably might be used to great advantage.

In conclusion it is pointed out that the location of such masses as engines, landing gears, and fuel tanks might be used to advantage in increasing the flutter speed of a given wing. As shown by the particular problem analyzed herein and by other experiences a definite region exists, peculiar to a given wing, in which masses added forward of the elastic axis of the wing tend to increase the flutter speed of the wing.

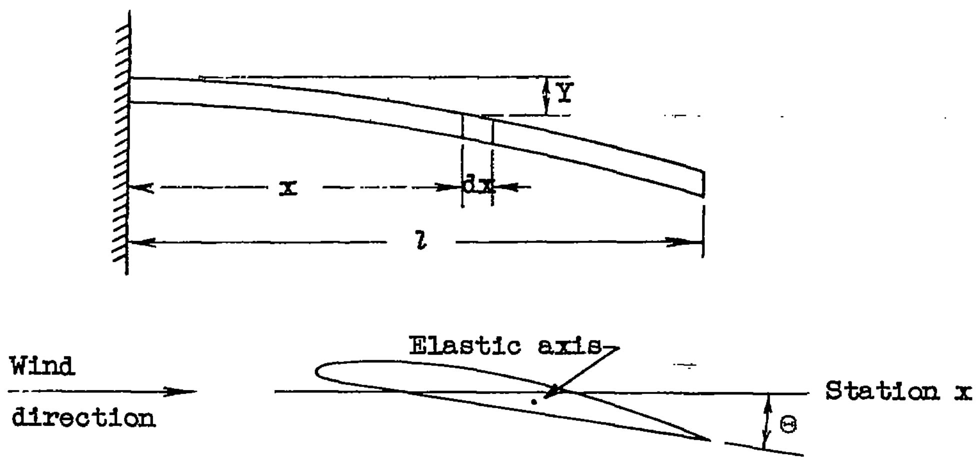
Langley Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Field, Va., November 30, 1948

APPENDIX A

OUTLINE AND EXTENSION OF METHODS OF FLUTTER ANALYSIS AS PRESENTED IN REFERENCES 3 AND 4

1. Derivation of the Differential Equations That Govern the Motion of a Wing at Flutter Speed

Consider a spanwise element of incremental length dx at station x of a wing oscillating in bending and torsion in a free stream of fluid (see sketch).



The displacements Y and Θ of an element of the elastic axis are functions of x and t. In order that this element remain in dynamic equilibrium the external forces and moments on the element must balance the inertia forces and moments.

The external forces and moments consist of transverse shearing forces and torsional moments, which are transmitted from one element of the wing to the next, plus the aerodynamic lift force and pitching moment and internal or structural damping. Structural damping is not taken into consideration in this discussion, although its inclusion would add no computational difficulties.

The transverse shearing force acting upward at x is

$$N = -EI_b \frac{\partial^3 Y}{\partial^3 x} \tag{A1}$$

and that acting downward at $(x + dx)$ is

$$N + \frac{\partial N}{\partial x} dx = -EI_b \frac{\partial^3 Y}{\partial x^3} - EI_b \frac{\partial^4 Y}{\partial x^4} dx \tag{A2}$$

Similarily the nose-down torsional moment acting at x is

$$T = GJ \frac{\partial \Theta}{\partial x} \tag{A3}$$

and at $(x + dx)$ the nose-up torsional moment is

$$T + \frac{\partial T}{\partial x} dx = GJ \frac{\partial \Theta}{\partial x} + GJ \frac{\partial^2 \Theta}{\partial x^2} dx \tag{A4}$$

The two-dimensional aerodynamic forces acting on an element dx of an oscillating airfoil have been derived by Theodorsen (reference 8) and can be written as a lift force and aerodynamic moment acting about the elastic axis of the wing, respectively, as

$$L\ dx = \left(\omega^2 L_y Y + \omega L_y' \frac{\partial Y}{\partial t} + \omega^2 L_\theta \Theta + \omega L_\theta' \frac{\partial \Theta}{\partial t}\right) dx \tag{A5}$$

$$M\ dx = \left(\omega^2 M_y Y + \omega M_y' \frac{\partial Y}{\partial t} + \omega^2 M_\theta \Theta + \omega M_\theta' \frac{\partial \Theta}{\partial t}\right) dx \tag{A6}$$

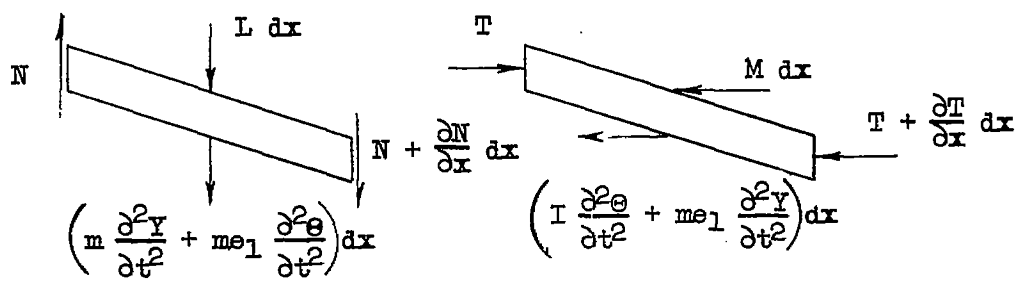
The inertia force of the element dx can be written

$$\left(m \frac{\partial^2 Y}{\partial t^2} + me_1 \frac{\partial^2 \Theta}{\partial t^2}\right) dx \tag{A7}$$

and the inertia moment as

$$\left(I \frac{\partial^2 \Theta}{\partial t^2} + me_1 \frac{\partial^2 Y}{\partial t^2}\right) dx \tag{A8}$$

Diagrams of the forces and moments acting on an element of wing of length dx at station x are as follows:



Imposing the conditions of dynamic equilibrium of the element at x by equating inertia forces to external forces and inertia moments to external moments gives the two differential equations that govern the motion of the wing:

$$\left.\begin{aligned} m\,\frac{\partial^2 Y}{\partial t^2} + me_1\,\frac{\partial^2 \Theta}{\partial t^2} &= -EI_b\,\frac{\partial^4 Y}{\partial x^4} + \omega^2 L_y Y + \omega L_y{}'\,\frac{\partial Y}{\partial t} + \omega^2 L_\theta \Theta + \omega L_\theta{}'\,\frac{\partial \Theta}{\partial t} \\ I\,\frac{\partial^2 \Theta}{\partial t^2} + me_1\,\frac{\partial^2 Y}{\partial t^2} &= GJ\,\frac{\partial^2 \Theta}{\partial x^2} + \omega^2 M_y Y + \omega M_y{}'\,\frac{\partial Y}{\partial t} + \omega^2 M_\theta \Theta + \omega M_\theta{}'\,\frac{\partial \Theta}{\partial t} \end{aligned}\right\} \qquad \text{(A9)}$$

2. Boundary Conditions for a Uniform Cantilever Wing Carrying an Arbitrarily Placed Weight at Flutter Speed

The boundary conditions that must be imposed upon equations (A9) for a uniform cantilever wing are

(1) $Y(0,t) = 0$

(2) $EI_b\left[\frac{\partial}{\partial x}\,Y'(x,t)\right]_{x=0} = 0$

(3) $\theta(0,t) = 0$

(4) $EI_b\left[\frac{\partial^2}{\partial x^2} Y(x,t)\right]_{x=l} = 0$

(5) $EI_b\left[\frac{\partial^3}{\partial x^3} Y(x,t)\right]_{x=l} = 0$

(6) $GJ\left[\frac{\partial}{\partial x} \theta(x,t)\right]_{x=l} = 0$

These are the usual conditions that must be imposed on a vibrating cantilever beam. Condition (1) is the condition that the end at $x = 0$ is supported (either hinged or built in). Conditions (2) and (3) imply that this end is fixed or built in. Conditions (4), (5), and (6) imply, respectively, that there is no bending moment, transverse shearing force, or torsional moment acting at the tip $x = l$.

If there is an arbitrarily placed weight on the wing, other conditions must be imposed that will determine the effect of the weight upon the motion of the wing. If the weight is considered as concentrated at some point on the chord line at station $x = l_1$, it will create discontinuities in both transverse shear and torsional moment. The magnitude of these discontinuities are known functions of the mass of the weight, the location of the weight, and the acceleration of the wing. The remaining conditions required to complete the boundary-value problem for the general motion of the weighted wing are, therefore,

(7) $$EI_b\left\{\left[\frac{\partial^3}{\partial x^3} Y(x,t)\right]_{x=(l_1-0)} - \left[\frac{\partial^3}{\partial x^3} Y(x,t)\right]_{x=(l_1+0)}\right\}$$

$$= \frac{W_w}{g}\left[\frac{\partial^2}{\partial t^2} Y(x,t) + e_2 \frac{\partial^2}{\partial t^2} \theta(x,t)\right]_{x=l_1}$$

(8) $$GJ\left\{\left[\frac{\partial}{\partial x} \theta(x,t)\right]_{x=(l_1-0)} - \left[\frac{\partial}{\partial x} \theta(x,t)\right]_{x=(l_1+0)}\right\}$$

$$= -\frac{W_w}{g}\left[e_2 \frac{\partial^2}{\partial t^2} Y(x,t) + K_2{}^2 \frac{\partial^2}{\partial t^2} \theta(x,t)\right]_{x=l_1}$$

For the purpose of flutter analysis it is assumed that the motions in both bending and torsion are harmonic and that the frequencies in bending and torsion are equal. Therefore, only the particular form that the solution to the boundary-value problem has when these conditions obtain need be sought. These conditions imply that $Y(x,t)$ and $\Theta(x,t)$ are of the forms

$$\left.\begin{aligned} Y(x,t) &= y(x)e^{i\omega t} \\ \Theta(x,t) &= \theta(x)e^{i\omega t} \end{aligned}\right\} \tag{A10}$$

where, on the right-hand side of equations (A10), y and θ are now complex amplitude functions of the span coordinate x from which the shape and phase relation of the wing at any fixed time during flutter can be obtained.

If the values of Y and Θ from equations (A10) are substituted into both differential equations (A9) and into the boundary conditions, the problem is greatly simplified. The differential equations become independent of t and appear as ordinary differential equations with constant coefficients. After making the substitution and rearranging terms, the equations of motion can be written as

$$\left.\begin{aligned} EI_b \frac{d^4y}{dx^4} - \left(m + L_y + iL_y'\right)\omega^2 y - \left(me_1 + L_\theta + iL_\theta'\right)\omega^2\theta &= 0 \\ GJ \frac{d^2\theta}{dx^2} + \left(me_1 + M_y + iM_y'\right)\omega^2 y + \left(I + M_\theta + iM_\theta'\right)\omega^2\theta &= 0 \end{aligned}\right\} \tag{A11}$$

or more simply as

$$\left.\begin{aligned} \frac{d^4y}{dx^4} - \alpha y - \beta\theta &= 0 \\ \frac{d^2\theta}{dx^2} + \gamma y + \delta\theta &= 0 \end{aligned}\right\} \tag{A12}$$

The boundary conditions also become independent of t and can be written as follows:

(1') $y(0) = 0$

(2') $y'(0) = 0$

(3') $\theta(0) = 0$

(4') $y''(l) = 0$

(5') $y'''(l) = 0$

(6') $\theta'(l) = 0$

$$(7')\qquad EI_b\left[y'''(l_1 - 0) - y'''(l_1 + 0)\right] = -\frac{W_w}{g}\omega^2\left[y(l_1) + e_2\theta(l_1)\right]$$

$$(8')\qquad GJ\left[\theta'(l_1 - 0) - \theta'(l_1 + 0)\right] = \frac{W_w}{g}\omega^2\left[e_2y(l_1) + K_2{}^2\theta(l_1)\right]$$

3. Solution of Boundary-Value Problems in Ordinary Differential Equations by Operational Methods and Application to a Beam Carrying an Arbitrarily Placed Weight

The boundary-value problem given by equations (A12) and conditions (1') to (8') can be solved by straightforward methods of solving ordinary differential equations with constant coefficients. The operational method, however, is a much easier and shorter approach, particularly in view of the discontinuities in shear and torque.

Briefly, the solution of a boundary-value problem by operational methods consists of applying the Laplace transform to the differential equations, the initial conditions (root conditions when applied to beam problems), and certain forms of other boundary conditions; of solving the resulting system for the transform of each dependent variable; and then by applying the inversion integral to the results. The remaining boundary conditions are then used to set up relations among whatever undetermined parameters that might remain.

In the case of flutter analysis a complete solution to the equations is not needed but only the conditions under which an unstable equilibrium may exist. The relations that can be set up between the undetermined parameters correspond precisely to this condition. In other words these relations appear as a system of homogeneous equations and the satisfaction of the condition that this system of equations have a common solution other than the trivial solution corresponds to the border-line condition separating the damped and undamped oscillations of the wing.

The Laplace transform of $f(x)$ is

$$L\left\{f(x)\right\} = \int_0^\infty e^{-sx} f(x)\, dx = \bar{f}(s) \tag{A13}$$

where s may be real or complex and $x > 0$. The sufficient conditions that this infinite integral exist are that $f(x)$ have no infinite discontinuities for $x \geq 0$ and that $f(x)$ be of exponential order as $x \longrightarrow \infty$. (See reference 9.) In other words finite discontinuities such as those appearing in the foregoing problem do not invalidate the operational approach.

The Laplace transform of the nth derivative of a continuous function with continuous derivatives, for which the function and all its derivatives are of exponential order, can be obtained directly from equation (A13) as

$$L\left\{f^n(x)\right\} = s^n\bar{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdot \cdot \cdot \cdot - f^{n-1}(0) \tag{A14}$$

The Laplace transform is linear in the same sense as differentiation or integration. That is, if a_i and b_i are constants

$$L\left\{a_nf^n(x) + a_{n-1}f^{n-1}(x) + \cdot \cdot \cdot + a_of(x) + b_m\theta^m(x) + \cdot \cdot \cdot + b_o\theta(x)\right\}$$

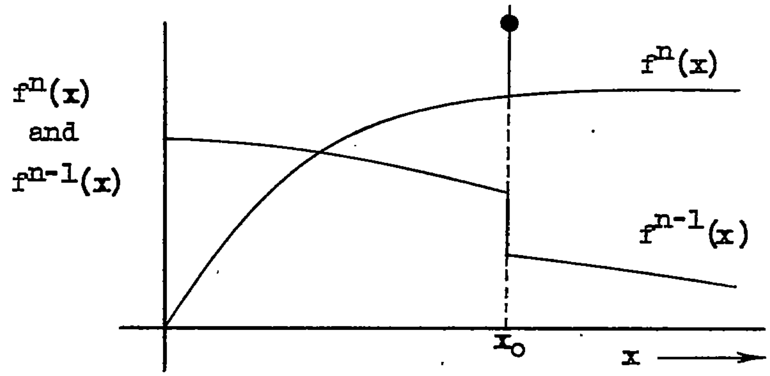
$$= a_nL\left\{f^n(x)\right\} + a_{n-1}L\left\{f^{n-1}(x)\right\} + \cdot \cdot \cdot + a_oL\left\{f(x)\right\}$$

$$+ b_mL\left\{\theta^m(x)\right\} + \cdot \cdot \cdot + b_oL\left\{\theta(x)\right\} \tag{A15}$$

Thus the Laplace transform of a linear differential equation with constant coefficients is generally a sum of expressions similar to equation (A14).

In equation (A14) the quantities $f(0)$, $f'(0)$, . . . , $f^{n-1}(0)$ are the boundary conditions at the origin of the dependent variable (wing root) that corresponds to constants of integration. When these quantities are given they are put directly into the transformed equation. When the quantities are not given they correspond to what has been called undetermined parameters in the preceding paragraphs and must later be determined in terms of other boundary conditions.

Finite discontinuities in a function or any of its derivatives are taken into account by proper attention to the limiting values that the function or its derivatives have on the two sides of the discontinuity. In particular, if a function and its first n derivatives are of exponential order, if the first $(n - 2)$ derivatives are continuous, if the $(n - 1)^{st}$ derivative has a finite discontinuity at x_o, and if the nth derivative is continuous except for a singular point at x_o, (see sketch)



the Laplace transform of the nth derivative has the form

$$L\left\{f^n(x)\right\} = s^n\bar{f}(s) - s^{n-1}f(0) - . . .$$

$$- sf^{n-2}(0) - e^{-sx_o}\left[f^{n-1}(x_o + 0) - f^{n-1}(x_o - 0)\right] \qquad \text{(A16)}$$

where $f(x_o + 0)$ is the value of $f(x)$ as x approaches x_o from the right and $f(x_o - 0)$ is the value of $f(x)$ as x approaches x_o from the left. In other words the terms in the brackets express the magnitude of the discontinuity in $f^{n-1}(x)$ at x_o in the $(n - 1)^{st}$ derivative at x_o.

An examination of the boundary-value problem, equation (A12), shows that the transform will be given by a sum of expressions precisely of the form of equation (A16).

In order to interpret the transformed function $\bar{f}(s)$ in terms of the original function $f(x)$, use may be made of the inversion integral discussed in text books on operational calculus; or one may refer directly to tables of transform.

As a simple example the operational method is applied to a cantilever beam carrying an arbitrarily placed weight and assumed to be vibrating in a vacuum in bending only.

The boundary-value problem for this case can be written

$$EI_b \frac{d^4 y}{dx^4} = m\omega^2 y \tag{A17}$$

$$\left.\begin{aligned}
&\text{(a)} \qquad y(0) = y'(0) = 0 \\
&\text{(b)} \qquad y''(l) = y'''(l) = 0 \\
&\text{(c)} \qquad EI_b\left[y'''(l_1 - 0) - y'''(l_1 + 0)\right] = -\frac{W_w\omega^2}{g}\, y(l_1)
\end{aligned}\right\} \tag{A18}$$

where the symbols have the same meaning as in equation (A12).

If the root conditions (a) and the boundary condition (c) are used, the transformed problem solved for $\bar{y}(s)$ gives

$$\bar{y}(s) = \frac{sY_2}{s^4 - \alpha^4} + \frac{Y_3}{s^4 - \alpha^4} + \frac{W_w\omega^2}{gEI_b}\,\frac{y(l_1)}{s^4 - \alpha^4}\, e^{-sl_1} \tag{A19}$$

where, for brevity, $Y_2 = y''(0)$, $Y_3 = y'''(0)$, and $\alpha^4 = \frac{m\omega^2}{EI_b}$.

The inverse transform of equation (A19) is (see pair nos. 31 and 32, p. 296, and relation 12, p. 294, of reference 9)

$$y(x) = \frac{Y_2}{2\alpha^2}\left(\cosh \alpha x - \cos \alpha x\right) + \frac{Y_3}{2\alpha^3}\left(\sinh \alpha x - \sin \alpha x\right)$$

$$+ \frac{W_w\omega^2}{2\alpha^3 gEI_b} y(l_1)\left[\sinh \alpha(x - l_1) - \sin \alpha(x - l_1)\right] \tag{A20}$$

or

$$y(x) = \frac{Y_2}{2\alpha^2}\left(\cosh \alpha x - \cos \alpha x\right) + \frac{Y_3}{2\alpha^3}\left(\sinh \alpha x - \sin \alpha x\right)$$

$$+ \frac{W_w\omega^2}{2\alpha^3 gEI_b}\left[\frac{Y_2}{2\alpha^2}\left(\cosh \alpha l_1 - \cos \alpha l_1\right)\right.$$

$$\left. + \frac{Y_3}{2\alpha^3}\left(\sinh \alpha l_1 - \sin \alpha l_1\right)\right]\left[\sinh \alpha(x - l_1)\right.$$

$$\left. - \sin \alpha(x - l_1)\right] \tag{A21}$$

where the last bracket is zero when $x - l_1 \leqq 0$.

Imposing boundary conditions (b) gives two homogeneous equations in Y_2 and Y_3. Each value of α that will cause the determinant of the coefficients of Y_2 and Y_3 to vanish corresponds to a mode of vibration.

This result has been applied to the wing and weight discussed in the text of this paper with the weight located 17 inches from the root. The deflection and shear curves due to the first uncoupled modes in bending only have been computed and are plotted in figure 6. Corresponding results have been computed by a 20-station process of iteration discussed in reference 12 and plotted in the same figure.

4. Representation of the Inverse Transform of the Boundary-Value Problem, Equation (A12), by a Power Series

The transform of both $y(x)$ and $\theta(x)$ of equation (A12) are of the form

$$f(s) = \frac{P_1(s)}{q(s)} + \frac{P_2(s)}{q(s)} e^{-x_0 s} \tag{A22}$$

where $P_1(s)$ and $P_2(s)$ are polynomials both of lower degree than $q(s)$. Neither $P_1(s)$ or $P_2(s)$ have common factors with $q(s)$ where $q(s)$ is of the specific form

$$q(s) = s^6 + as^4 + bs^2 + c = \left(s^2 - R_1\right)\left(s^2 - R_2\right)\left(s^2 - R_3\right) \tag{A23}$$

where the coefficients a, b, and c and the roots squared R_1, R_2, and R_3 are complex. The inverse function associated with such a transform gives $f(x)$ in terms of circular and hyperbolic functions of $x\sqrt{R_i}$, but with the results in this form the process of solving the flutter determinant becomes very cumbersome.

By making use of the properties of symmetric functions, Goland and Luke (reference 4) outlined a simple method of obtaining series expansions for the transforms of equations (A12) that does not involve the meticulous task of finding the roots of $q(s)$. The inversions of these expansions give $y(x)$ and $\theta(x)$ in the form of convergent series.

For the development of these series it is first necessary to consider $q(s)$ as a cubic in s^2; namely,

$$q(s) = \prod_{i=1}^{3}\left(s^2 - R_i\right) = s^6 \prod_{i=1}^{3}\left(1 - \frac{R_i}{s^2}\right) \tag{A24}$$

By making use of the binomial theorem, $1/q(s)$ can be written as

$$\frac{1}{q(s)} = \frac{1}{s^6} \prod_{i=1}^{3}\left(1 + \frac{R_i}{s^2} + \frac{R_i^2}{s^4} + \frac{R_i^3}{s^6} + \ldots\right) \tag{A25}$$

Equation (A25) is independent of any interchange of the parameters R_1, R_2, and R_3 and thus satisfies the description of a symmetric function in these parameters. (For a discussion of symmetric functions see reference 10 or any text on higher algebra or theory of equations.) If the indicated multiplication in equation (A25) is carried out, the results can be written

$$\frac{1}{q(s)} = \frac{1}{s^6}\left(T_0 + \frac{T_1}{s^2} + \frac{T_2}{s^4} + \ldots + \frac{T_n}{s^{2n}} + \ldots\right) \tag{A26}$$

where the general term T_n represents the sum of all possible symmetric polynomials in R_1, R_2, and R_3 which are of degree n and with all coefficients unity. By making use of Newton's identity relative to symmetric polynomials, that is

$$T_n = -aT_{n-1} - bT_{n-2} - cT_{n-3} \tag{A27}$$

where the value of any T_{n-j} is to be disregarded when $n - j < 0$, every T_n can be written in terms of the coefficients a, b, and c of equation (A23); for example,

$$\left.\begin{aligned}
T_0 &= 1 \\
T_1 &= -a \\
T_2 &= a^2 - b \\
T_3 &= -a^3 + 2ab - c \\
&\ldots\ldots
\end{aligned}\right\} \tag{A28}$$

With the aid of equation (A26) and equations (9) and (10) of the text the inverse transform of equation (A22) or of $\bar{y}(s)$ and $\bar{\theta}(s)$ can therefore be written as a sum of terms of the type given in equations (9) and (10) where the T_n's enter as coefficients in the numerator and are easily evaluated in terms of the coefficients of a known cubic equation. In the application to flutter analysis only the first few T_n's are usually necessary because the resulting series is generally found to be highly convergent.

APPENDIX B

DERIVATION OF THE FLUTTER DETERMINANT AND SAMPLE CALCULATIONS

Introduction

In this section the flutter determinant is formally derived and the method described in the text for solving the determinant is illustrated with sample calculations for a specific example. Also final expressions for the deflection curves are given from which amplitude and phase angle curves of deflection, shear, and torque are calculated for a specific case. The calculated amplitudes are compared with corresponding curves computed from the fundamental uncoupled modes in bending and torsion.

Derivation of the Flutter Determinant

In equations (11) and (12) of the text it is first necessary to evaluate the expressions

$$\left[y'''(l_1 - 0) - y'''(l_1 + 0)\right]$$

and

$$\left[\theta'(l_1 - 0) - \theta'(l_1 + 0)\right]$$

in terms of Y_2, Y_3, and θ_1. Since terms involving $(x - l_1)$ drop out of both equation (11) and equation (12) for $x = l_1$, the values of $y(l_1)$

and $\theta(l_1)$ can be obtained directly from these equations. The values of $y(l_1)$ and $\theta(l_1)$ substituted into conditions (c) and (d) of the text give the desired relations; namely,

$$y'''(l_1 - 0) - y'''(l_1 + 0) = -\frac{W_w\omega^2}{EI_b g}\left[y(l_1) + e_2\theta(l_1)\right]$$

$$= -\frac{W_w\omega^2}{EI_b g}\left\{Y_2\left[(\delta - e_2\gamma)\sum_{n=0}^{\infty}\frac{T_n l_1^{2n+4}}{(2n + 4)!}\right.\right.$$

$$\left. + \sum_{n=0}^{\infty}\frac{T_n l_1^{2n+2}}{(2n + 2)!}\right] + Y_3\left[(\delta - e_2\gamma)\sum_{n=0}^{\infty}\frac{T_n l_1^{2n+5}}{(2n + 5)!}\right.$$

$$\left. + \sum_{n=0}^{\infty}\frac{T_n l_1^{2n+3}}{(2n + 3)!}\right]$$

$$+ \theta_1\left[(\beta - e_2\alpha)\sum_{n=0}^{\infty}\frac{T_n l_1^{2n+5}}{(2n + 5)!}\right.$$

$$\left.\left. + e_2\sum_{n=0}^{\infty}\frac{T_n l_1^{2n+1}}{(2n + 1)!}\right]\right\} \qquad \text{(B1)}$$

and

$$\theta'(l_1 - 0) - \theta'(l_1 + 0) = \frac{W_w\omega^2}{GJg}\left[e_2 y(l_1) + K_2{}^2\theta(l_1)\right]$$

$$= \frac{W_w\omega^2}{GJg}\left\{Y_2\left[(e_2\delta - K_2{}^2)\sum_{n=0}^{\infty}\frac{T_n l_1{}^{2n+4}}{(2n + 4)!}\right.\right.$$

$$\left. + e_2\sum_{n=0}^{\infty}\frac{T_n l_1{}^{2n+2}}{(2n + 2)!}\right]$$

$$+ Y_3\left[(e_2\delta - K_2{}^2\gamma)\sum_{n=0}^{\infty}\frac{T_n l_1{}^{2n+5}}{(2n + 5)!}\right.$$

$$\left. + e_2\sum_{n=0}^{\infty}\frac{T_n l_1{}^{2n+3}}{(2n + 3)!}\right]$$

$$+ \theta_1\left[(e_2\beta - K_2{}^2\alpha)\sum_{n=0}^{\infty}\frac{T_n l_1{}^{2n+5}}{(2n + 5)!}\right.$$

$$\left.\left. + K_2{}^2\sum_{n=0}^{\infty}\frac{T_n l_1{}^{2n+1}}{(2n + 1)!}\right]\right\} \tag{B2}$$

Substituting equations (B1) and (B2) into equations (11) and (12) gives

$$y(x) = h_1(x)Y_2 + h_2(x)Y_3 + h_3(x)\theta_1 \tag{B3}$$

$$\theta(x) = g_1(x)Y_2 + g_2(x)Y_3 + g_3(x)\theta_1 \tag{B4}$$

where

$$h_1(x) = \sum_{n=0}^{\infty} \frac{T_n x^{2n+2}}{(2n + 2)!} + \delta \sum_{n=0}^{\infty} \frac{T_n x^{2n+4}}{(2n + 4)!} + \frac{W_w \omega^2}{EI_b g}\left[(\delta - e_2\gamma) \sum_{n=0}^{\infty} \frac{T_n l_1^{2n+4}}{(2n + 4)!}\right.$$

$$\left. + \sum_{n=0}^{\infty} \frac{T_n l_1^{2n+2}}{(2n + 2)!}\right]\left[\delta \sum_{n=0}^{\infty} \frac{T_n\left(x - l_1\right)^{2n+5}}{(2n + 5)!} + \sum_{n=0}^{\infty} \frac{T_n\left(x - l_1\right)^{2n+3}}{(2n + 3)!}\right]$$

$$- \frac{\beta W_w \omega^2}{GJg}\left[(e_2\delta - K_2^2) \sum_{n=0}^{\infty} \frac{T_n l_1^{2n+4}}{(2n + 4)!}\right.$$

$$\left. + e_2 \sum_{n=0}^{\infty} \frac{T_n l_1^{2n+2}}{(2n + 2)!}\right] \sum_{n=0}^{\infty} \frac{T_n\left(x - l_1\right)^{2n+5}}{(2n + 5)!}$$

$$h_2(x) = \delta \sum_{n=0}^{\infty} \frac{T_n x^{2n+5}}{(2n + 5)!} + \sum_{n=0}^{\infty} \frac{T_n x^{2n+3}}{(2n + 3)!} + \frac{W_w \omega^2}{EI_b g}\left[(\delta - e_2\gamma) \sum_{n=0}^{\infty} \frac{T_n l_1^{2n+5}}{(2n + 5)!}\right.$$

$$\left. + \sum_{n=0}^{\infty} \frac{T_n l_1^{2n+3}}{(2n + 3)!}\right]\left[\delta \sum_{n=0}^{\infty} \frac{T_n\left(x - l_1\right)^{2n+5}}{(2n + 5)!} + \sum_{n=0}^{\infty} \frac{T_n\left(x - l_1\right)^{2n+3}}{(2n + 3)!}\right]$$

$$- \frac{\beta W_w \omega^2}{GJg}\left[(e_2\delta - K_2^2\gamma) \sum_{n=0}^{\infty} \frac{T_n l_1^{2n+5}}{(2n + 5)!}\right.$$

$$\left. + e_2 \sum_{n=0}^{\infty} \frac{T_n l_1^{2n+3}}{(2n + 3)!}\right] \sum_{n=0}^{\infty} \frac{T_n\left(x - l_1\right)^{2n+5}}{(2n + 5)!}$$

$$h_3(x) = \beta\sum_{n=0}^{\infty}\frac{T_n x^{2n+5}}{(2n+5)!} + \frac{W_w\omega^2}{EI_b g}\left[(\beta - e_2\alpha)\sum_{n=0}^{\infty}\frac{T_n l_1^{2n+5}}{(2n+5)!}\right.$$

$$\left. + e_2\sum_{n=0}^{\infty}\frac{T_n l_1^{2n+1}}{(2n+1)!}\right]\left[\delta\sum_{n=0}^{\infty}\frac{T_n\left(x - l_1\right)^{2n+5}}{(2n+5)!} + \sum_{n=0}^{\infty}\frac{T_n\left(x - l_1\right)^{2n+3}}{(2n+3)!}\right]$$

$$- \frac{\beta W_w\omega^2}{GJg}\left[(e_2\beta - K_2^2\alpha)\sum_{n=0}^{\infty}\frac{T_n l_1^{2n+5}}{(2n+5)!}\right.$$

$$\left. + K_2^2\sum_{n=0}^{\infty}\frac{T_n l_1^{2n+1}}{(2n+1)!}\right]\sum_{n=0}^{\infty}\frac{T_n\left(x - l_1\right)^{2n+5}}{(2n+5)!}$$

$$g_1(x) = -\gamma\sum_{n=0}^{\infty}\frac{T_n x^{2n+4}}{(2n+4)!} + \frac{W_w\omega^2}{GJg}\left[(e_2\delta - K_2^2)\sum_{n=0}^{\infty}\frac{T_n l_1^{2n+4}}{(2n+4)!}\right.$$

$$\left. + e_2\sum_{n=0}^{\infty}\frac{T_n l_1^{2n+2}}{(2n+2)!}\right]\left[\alpha\sum_{n=0}^{\infty}\frac{T_n\left(x - l_1\right)^{2n+5}}{(2n+5)!} - \sum_{n=0}^{\infty}\frac{T_n\left(x - l_1\right)^{2n+1}}{(2n+1)!}\right]$$

$$- \frac{W_w\omega^2}{EI_b g}\left[(\delta - e_2\gamma)\sum_{n=0}^{\infty}\frac{T_n l_1^{2n+4}}{(2n+4)!} + \sum_{n=0}^{\infty}\frac{T_n l_1^{2n+2}}{(2n+2)!}\right]\sum_{n=0}^{\infty}\frac{T_n\left(x - l_1\right)^{2n+5}}{(2n+5)!}$$

$$g_2(x) = -\gamma\sum_{n=0}^{\infty}\frac{T_n x^{2n+5}}{(2n+5)!} + \frac{W_w\omega^2}{GJg}\left[(e_2\delta - K_2^2\gamma)\sum_{n=0}^{\infty}\frac{T_n l_1^{2n+5}}{(2n+5)!}\right.$$

$$\left. + e_2\sum_{n=0}^{\infty}\frac{T_n l_1^{2n+3}}{(2n+3)!}\right]\left[\alpha\sum_{n=0}^{\infty}\frac{T_n\left(x - l_1\right)^{2n+5}}{(2n+5)!} - \sum_{n=0}^{\infty}\frac{T_n\left(x - l_1\right)^{2n+1}}{(2n+1)!}\right]$$

$$- \frac{W_w\omega^2\gamma}{EI_b g}\left[(\delta - e_2\gamma)\sum_{n=0}^{\infty}\frac{T_n l_1^{2n+5}}{(2n+5)!} + \sum_{n=0}^{\infty}\frac{T_n l_1^{2n+3}}{(2n+3)!}\right]\sum_{n=0}^{\infty}\frac{T_n\left(x - l_1\right)^{2n+5}}{(2n+5)!}$$

and

$$g_3(x) = \sum_{n=0}^{\infty} \frac{T_n x^{2n+1}}{(2n + 1)!} - \alpha \sum_{n=0}^{\infty} \frac{T_n x^{2n+5}}{(2n + 5)!} + \frac{W_w \omega^2}{GJg} \left[(e_2 \beta - K_2^2 \alpha) \sum_{n=0}^{\infty} \frac{T_n l_1^{2n+5}}{(2n + 5)!} \right.$$

$$\left. + K_2^2 \sum_{n=0}^{\infty} \frac{T_n l_1^{2n+1}}{(2n + 1)!} \right] \left[\alpha \sum_{n=0}^{\infty} \frac{T_n (x - l_1)^{2n+5}}{(2n + 5)!} - \sum_{n=0}^{\infty} \frac{T_n (x - l_1)^{2n+1}}{(2n + 1)!} \right]$$

$$- \frac{W_w \omega^2 \gamma}{EI_b g} \left[(\beta - e_2 \alpha) \sum_{n=0}^{\infty} \frac{T_n l_1^{2n+5}}{(2n + 5)!} \right.$$

$$\left. + e_2 \sum_{n=0}^{\infty} \frac{T_n l_1^{2n+1}}{(2n + 1)!} \right] \sum_{n=0}^{\infty} \frac{T_n (x - l_1)^{2n+5}}{(2n + 5)!}$$

By imposing conditions (b) of the text

$$y''(l) = y'''(l) = \theta'(l) = 0$$

upon equations (B3) and (B4), three equations are obtained (written in the text as equation (13)):

$$A_i Y_2 + B_i Y_3 + C_i \theta_1 = 0$$

where $i = 1, 2, \text{ and } 3$ and

$$A_1 = h_1''(l) \qquad B_1 = h_2''(l) \qquad C_1 = h_3''(l)$$

$$A_2 = h_1'''(l) \qquad B_2 = h_2'''(l) \qquad C_2 = h_3'''(l)$$

$$A_3 = g_1'(l) \qquad B_3 = g_2'(l) \qquad C_3 = g_3'(l)$$

Imposing the condition that the equations (13) have a solution other than the trivial solution $Y_2 = Y_3 = \theta_1 = 0$ results in the flutter determinant

$$\Delta = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = 0 \qquad \text{(B5)}$$

Sample Calculation of Flutter Speed

and Deflection Curves

A method of solving the flutter determinant given in the text is illustrated here by the solution of the determinant for the wing-weight combination discussed in the text when the spanwise location of the weight is 17 inches from the root. The values of $\frac{V}{b\omega} = \frac{1}{k}$ that are chosen are in the neighborhood of the experimental value and have available tabulated values of Theodorsen's function $C(k) = F + iG$.

Table I shows the actual computations required to evaluate the coefficients A_i, B_i, and C_i for $\frac{V}{b\omega} = 7.1429$ $(k = 0.14)$ and two values of $\frac{\omega}{2\pi} = f$ $(f = 25$ cps and $f = 28$ cps). From columns 32, 38, and 41 the determinant for $f = 25$ cps is

$$\Delta = \begin{vmatrix} (14.9200 - 2.8574i) & (12.8320 - 2.0315i) & -(7.3286 - 0.60021i) \\ (11.8000 - 3.6695i) & (10.2970 - 2.8566i) & -(5.4711 - 0.93233i) \\ (0.17030 - 0.66134i) & -(0.09077 + 0.59341i) & -(0.41138 - 0.28864i) \end{vmatrix}$$

or

$$\Delta = 1.0326 - 0.6948i$$

Similarly, for $f = 28$ cps,

$$\Delta = \begin{vmatrix} (18.6380 - 3.81151i) & (15.0860 - 2.63991i) & -(9.1238 - 0.85433i) \\ (15.5930 - 5.09351i) & (13.0080 - 3.79461i) & -(7.1158 - 1.3988i) \\ -(0.04177 + 0.87098i) & -(0.23526 + 0.75948i) & -(0.51403 - 0.37017i) \end{vmatrix}$$

or

$$\Delta = -0.4029 - 0.0312i$$

The determinant was evaluated in this manner for the same value of $v/b\omega$ and several other values of f. The process was then repeated for $\frac{v}{b\omega} = 6.25$ and several values of f and for $\frac{v}{b\omega} = 5.00$ and several values of f. The real and imaginary parts of the evaluated determinant for each value of $v/b\omega$ and the corresponding values of f are separately plotted in figure 7. The ordinates of the intersections of the different pairs of curves of real and imaginary parts were scaled in figure 7 and plotted as Δ_e against both $v/b\omega$ and f in figure 8. The zero ordinates of these curves give the value of $v/b\omega \left(\frac{v}{b\omega} = 6.93\right)$ and the values of $f(f = 28.04$ cps$)$ for which the determinant vanishes. From these values the flutter speed is readily calculated to be

$$v = (b\omega)(6.93) = (2\pi bf)(6.93) = \frac{(2\pi)(28.04)(6.93)}{3} = 407 \text{ fps}$$

As pointed out in appendix A the deflection curves at any specified time are given by equations (A10)

$$Y(x,t) = y(x)e^{i\omega t} = y(x)(\cos \omega t + i \sin \omega t)$$

$$\Theta(x,t) = \theta(x)e^{i\omega t} = \theta(x)(\cos \omega t + i \sin \omega t)$$

where final forms of $y(x)$ and $\theta(x)$ are given by equations (B3) and (B4) and where, at least, the relative values of the undetermined coefficients Y_2, Y_3, and θ_1 in equations (B3) and (B4) must be known. If the set of values of $v/b\omega$ and ω that satisfy the flutter determinant is used to determine the coefficients A_1, B_1, and C_1 in equations (13), there is obtained a system of three homogeneous equations in the three unknowns Y_2, Y_3, and θ_1 that have solutions other than the trivial solutions $Y_2 = Y_3 = \theta_1 = 0$. If these equations are each divided through by any one of the unknowns, say Y_2, there is obtained a consistent system of three equations in the two ratios Y_1/Y_2 and θ_1/Y_2. Any two of the three equations can therefore be solved for these ratios. Consequently, equations (B3) and (B4) can be written with one undetermined parameter that appears as a factor in each equation. Furthermore, since the coefficients A_i, B_i, and C_i are complex numbers the ratios Y_1/Y_2 and θ_1/Y_2 are complex numbers and equations (B3) and (B4) contain complex coefficients. The real and imaginary parts of these equations can be separated and the equations written

$$\left.\begin{aligned} y(x) &= Y_2\left[y_1(x) + iy_2(x)\right] \\ \theta(x) &= Y_2\left[\theta_2(x) + i\theta_3(x)\right] \end{aligned}\right\} \tag{B6}$$

If these relations are substituted into equations (A10),

$$\left.\begin{aligned} Y(x,t) &= Y_2\left\{y_1(x)\cos\omega t - y_2(x)\sin\omega t + i\left[y_2(x)\cos\omega t + y_1(x)\sin\omega t\right]\right\} \\ \Theta(x,t) &= Y_2\left\{\theta_2(x)\cos\omega t - \theta_3(x)\sin\omega t + i\left[\theta_3(x)\cos\omega t + \theta_2(x)\sin\omega t\right]\right\} \end{aligned}\right\} \tag{B7}$$

or

$$\left.\begin{aligned} Y(x,t) &= Y_2\sqrt{\left[y_1(x)\right]^2 + \left[y_2(x)\right]^2}\left[\cos(\omega t + \varphi_1) + i\sin(\omega t + \varphi_1)\right] \\ \Theta(x,t) &= Y_2\sqrt{\left[\theta_2(x)\right]^2 + \left[\theta_3(x)\right]^2}\left[\cos(\omega t + \varphi_2) + i\sin(\omega t + \varphi_2)\right] \end{aligned}\right\} \tag{B8}$$

where

$$\varphi_1 = \tan^{-1}\frac{y_2(x)}{y_1(x)}$$

and

$$\varphi_2 = \tan^{-1}\frac{\theta_3(x)}{\theta_2(x)}$$

and where $\varphi_1 - \varphi_2$ represents the difference in phase angle between bending motion and torsion motions at x.

The real parts of equations (B8) are interpreted to mean the motions in bending and torsion taken in a positive sense. The imaginary parts can then be interpreted as representing these same motions with a phase shift of $\pi/2$ radians.

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TABLE I - SAMPLE CALCULATIONS OF COEFFICIENTS A_1, B_1, AND C_1

$[k = 0.14]$

(1) f (cps)	(2) α	(3) β	(4) γ	(5) δ	(6) $\sum_0 \frac{T_n l_1^{2n+2}}{(2n+2)!}$
25	$0.65247-0.20341i$	$-0.48470+0.02865i$	$0.03178+0.05169i$	$0.16724-0.03661i$	$0.98302+0.00375i$
28	$0.81846-0.25515i$	$-0.60800+0.03594i$	$0.03986+0.06485i$	$0.20984-0.04593i$	$0.97789+0.00468i$
	(7) $\sum \frac{T_n l_1^{2n+4}}{(2n+4)!}$	(8) $[(5)\times(7)+(6)]Y_2$	(9) $\sum \frac{T_n l_1^{2n+5}}{(2n+5)!}$	(10) $\sum \frac{T_n l_1^{2n+3}}{(2n+3)!}$	(11) $[(5)\times(9)+(10)]Y_3$
	$0.16622+0.00033i$	$(1.0108-0.00228i)Y_2$	$0.04721+0.00007i$	$0.46744+0.00127i$	$(0.47534-0.00046i)Y_3$
	$0.16582+0.00041i$	$(1.0127-0.00285i)Y_2$	$0.04713+0.00009i$	$0.46583+0.00157i$	$(0.46682-0.00058i)Y_3$
	(12) $[(3)\times(9)]\theta_1$	(13) $y(l_1)=(8)+(11)+(12)$	(14) $[-(4)\times(7)]Y_2$	(15) $[-(4)\times(9)]Y_3$	(16) $[-(2)\times(9)]\theta_1$
	$(-0.02289+0.00132i)\theta_1$	$(1.0108-0.00228i)Y_2$ $+(0.47534-0.00046i)Y_3$ $+(-0.02289+0.00132i)\theta_1$	$(-0.00527-0.00860i)Y_2$	$(-0.00149-0.00244i)Y_3$	$(-0.03082+0.00956i)\theta_1$
	$(-0.02866+0.00164i)\theta_1$	$(1.0127-0.00285i)Y_2$ $+(0.46682-0.00058i)Y_3$ $+(-0.02869+0.00164i)\theta_1$	$(-0.00658-0.01077i)Y_2$	$(-0.00187-0.00306i)Y_3$	$(-0.03859+0.01195i)\theta_1$
	(17) $\sum \frac{T_n l_1^{2n+1}}{(2n+1)!}$	(18) $[(16)+(17)]\theta_1$	(19) $\theta(l_1)=(14)+(15)+(18)$	(20) $(13)+e_2(19)$	(21) $\frac{W}{g}\frac{\omega^2}{EI}(20)$
	$1.3695+0.00727i$	$(1.3387+0.01683i)\theta_1$	$(-0.00527-0.00860i)Y_2$ $+(-0.00149-0.00244i)Y_3$ $+(1.3387+0.01683i)\theta_1$	$(1.0122+0.00007i)Y_2$ $+(0.47575+0.00021i)Y_3$ $+(-0.38791-0.00327i)\theta_1$	$(-2.5332-0.00017i)Y_2$ $+(-1.1907-0.00051i)Y_3$ $+(0.97082+0.00818i)\theta_1$
	$1.3579+0.00899i$	$(1.3193+0.02095i)\theta_1$	$(-0.00658-0.01077i)Y_2$ $+(-0.00187-0.00306i)Y_3$ $+(1.3193+0.02095i)\theta_1$	$(1.0145+0.00008i)Y_2$ $+(0.46733+0.00026i)Y_3$ $+(-0.38839-0.00407i)\theta_1$	$(-3.1848-0.00026i)Y_2$ $+(-1.4671-0.00081i)Y_3$ $+(1.2193+0.01278i)\theta_1$

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TABLE I - SAMPLE CALCULATIONS OF COEFFICIENTS A_i, B_i, AND C_i - Concluded.

(22)	(23)	(24)	(25)	(26)
e_2 (13) $+k_2^2$ (19)	$\frac{W}{g}\frac{\omega^2}{GJ}$ (22)	$\sum \frac{x_n l^{2n}}{2n!}$	$\sum \frac{x_n l^{2n+2}}{(2n+2)!}$	$\sum \frac{x_n l^{2n+3}}{(2n+3)!}$
(-0.27694-0.00056i)Y_2 +(-0.12982-0.00021i)Y_3 +(0.19038+0.00195i)θ_1	(-1.4062-0.02886i)Y_2 +(-0.66059-0.00107i)Y_3 +(0.96875+0.00995i)θ_1	7.0385-1.9729i	10.0478-0.77851i	11.424-0.35654i
(-0.27704-0.00070i)Y_2 +(-0.12755-0.00026i)Y_3 +(0.18928+0.00243i)θ_1	(-1.7683-0.00449i)Y_2 +(-0.81415-0.00168i)Y_3 +(1.2082+0.01553i)θ_1	8.7043-2.5064i	10.6050-0.97996i	11.636-0.44942i
(27)	(28)	(29)	(30)	(31)
$\sum \frac{x_n l^{2n+1}}{(2n+1)!}$	$\sum \frac{x_n (l-l_1)^{2n+3}}{(2n+3)!}$	(3)×(26)	(5)×(26)	$\sum \frac{x_n (l-l_1)^{2n+1}}{(2n+1)!}$
8.0051-1.3737i	2.8156+0.00321i	-1.3648+0.07913i	0.47114-0.10255i	2.7457-0.09258i
9.0919-1.7320i	2.8018+0.00375i	-1.7036+0.09843i	0.58810-0.12789i	2.7925-0.11692i
(32)	(33)	(34)	(35)	(36)
[(24)+(5)×(25)] Y_2 + [(29)+(27)] Y_3+ [(29)] θ_1 + [(3)×(28)×(23)] + [(5)×(21)×(26)] + [(21)×(31)] $=y''(l)$	$\sum \frac{x_n l^{2n-1}}{(2n-1)!}$	$\sum \frac{x_n (l-l_1)^{2n+2}}{(2n+2)!}$	(3)×(34)	(5)×(34)
(14.9200-2.8574i)Y_2 +(12.8320-2.0315i)Y_3 +(-7.3286+0.60021i)θ_1	7.0966-2.3234i	3.3038-0.01793i	-1.6008+0.10336i	0.55204-0.12396i
(18.6380-3.8115i)Y_2 +(15.0860-2.6399i)Y_3 +(-9.1238+0.85433i)θ_1	9.1581-3.0144i	3.2979-0.02306i	-2.0043+0.13256i	0.69097-0.15630i
(37)	(38)	(39)	(40)	(41)
$\sum \frac{x_n (l-l_1)^{2n}}{2n!}$	[(35)+(5)×(27)] Y_2 + [(5)×(25)+(27)] Y_3 + [(3)×(25)] θ_1+ [(21)×(37)] + [(3)×(23)×(34)] + [(5)×(21)×(34)] $=y'''(l)$	$\sum \frac{x_n l^{2n+4}}{(2n+4)!}$	$\sum \frac{x_n (l-l_1)^{2n+4}}{(2n+4)!}$	[-(4)×(26)] Y_2+[-(4)×(7)] Y_3 + [-(5)×(7)+(24)] θ_1 + [(4)×(21)×(40)] + [(2)×(23)×(40)] + [-(23)×(37)] $=\theta'(l)$
1.6846-0.25672i	(11.8000-3.6695i)Y_2 +(10.2970-2.8566i)Y_3 +(-5.4711+0.93233i)θ_1	10.807-0.12747i	1.8198+0.00476i	(0.17030-0.66134i)Y_2 +(-0.09077-0.59341i)Y_3 +(-0.41138+0.28864i)θ_1
1.8685-0.32229i	(15.5930-5.0935i)Y_2 +(13.0080-3.7946i)Y_3 +(-7.1158+1.3988i)θ_1	10.853-0.16140i	1.8110+0.00587i	(-0.04177-0.87098i)Y_2 +(-0.23526-0.75948i)Y_3 +(-0.51403+0.37017i)θ_1

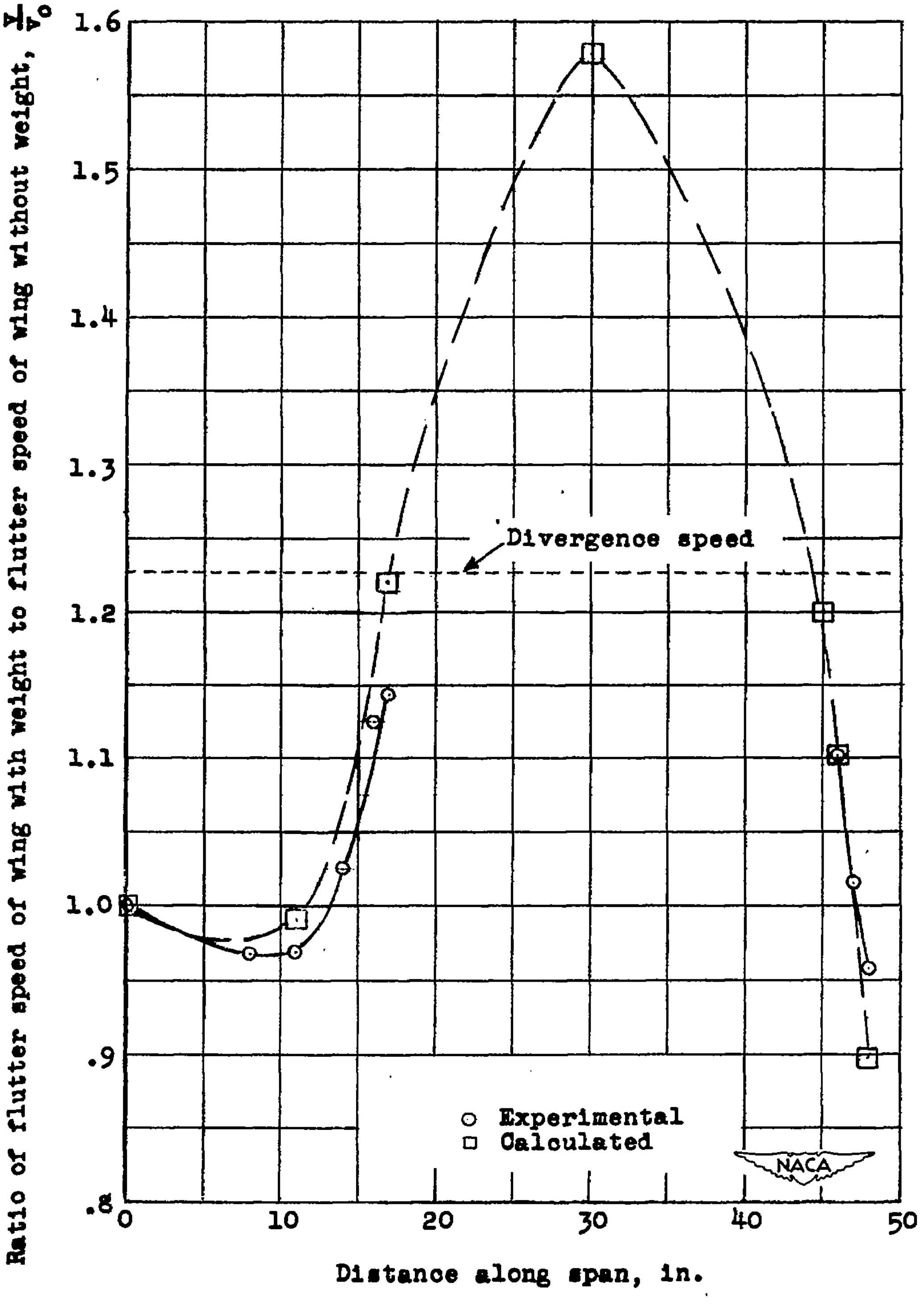


Figure 1.– Comparison of calculated and experimental flutter speeds for a particular wing–weight system.

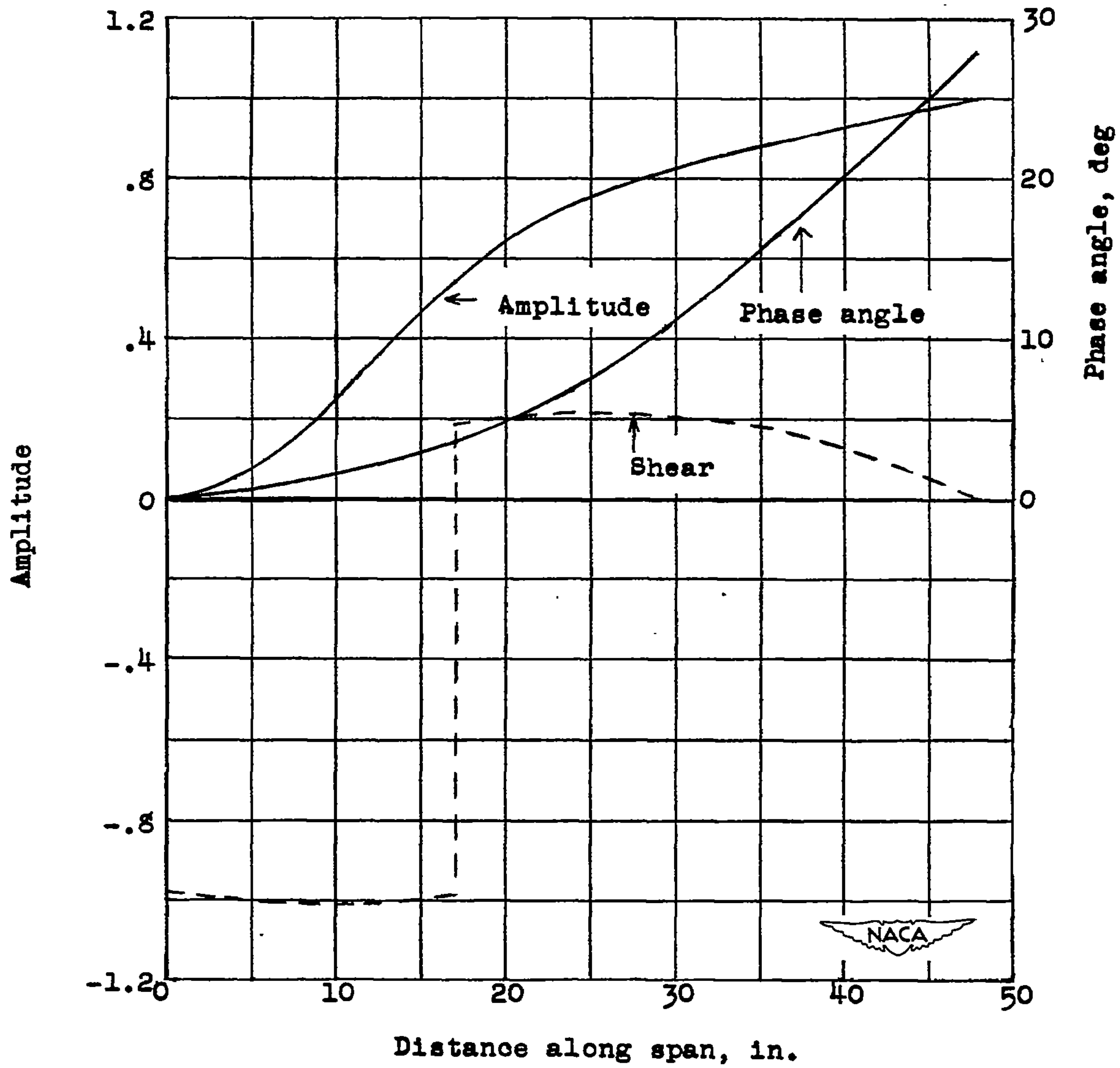


Figure 2.– Plot of amplitude and phase angle of displacement and shear curve in bending at flutter for $l_1 = 17$ inches (amplitude and shear referred to unit amplitude at tip in bending).

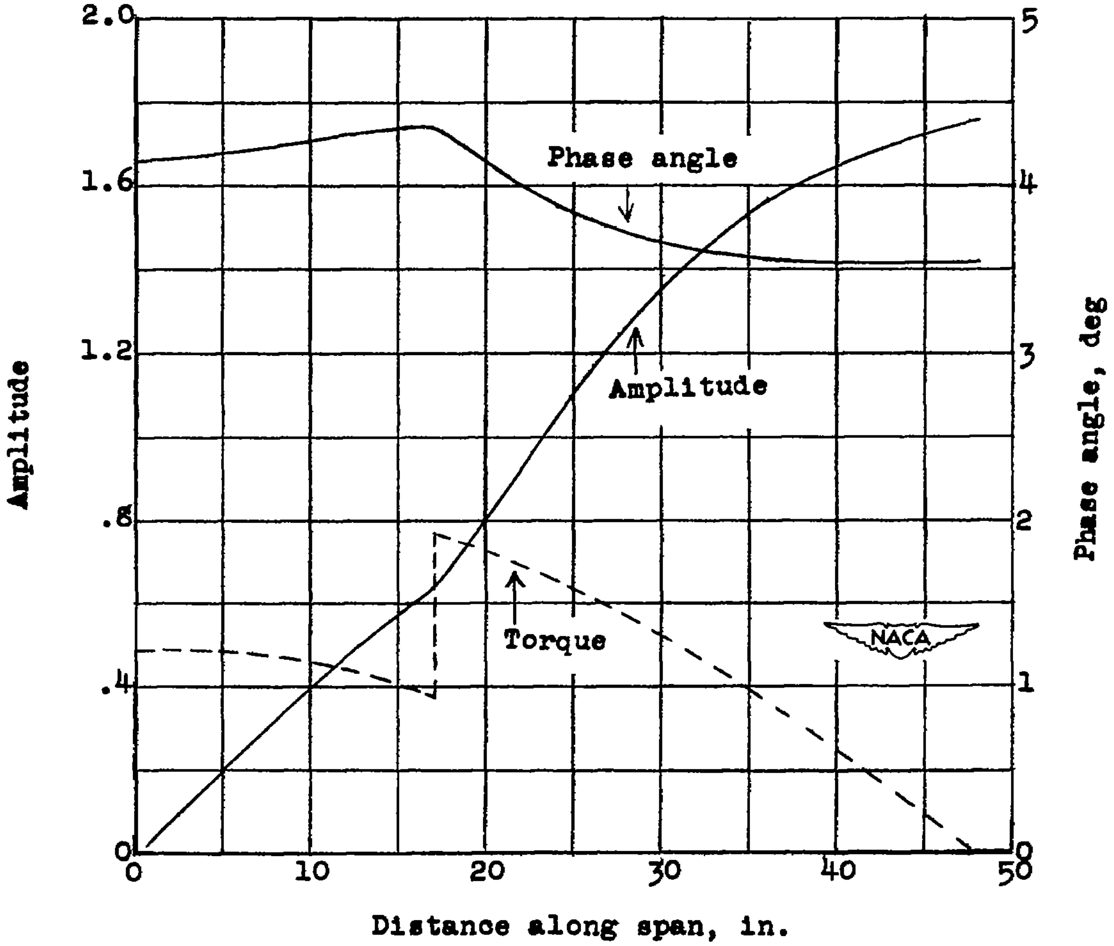


Figure 3.— Plot of amplitude and phase angle of torsional displacement and torque for $l_1 = 17$ inches at flutter (amplitude and torque referred to unit amplitude at tip in bending).

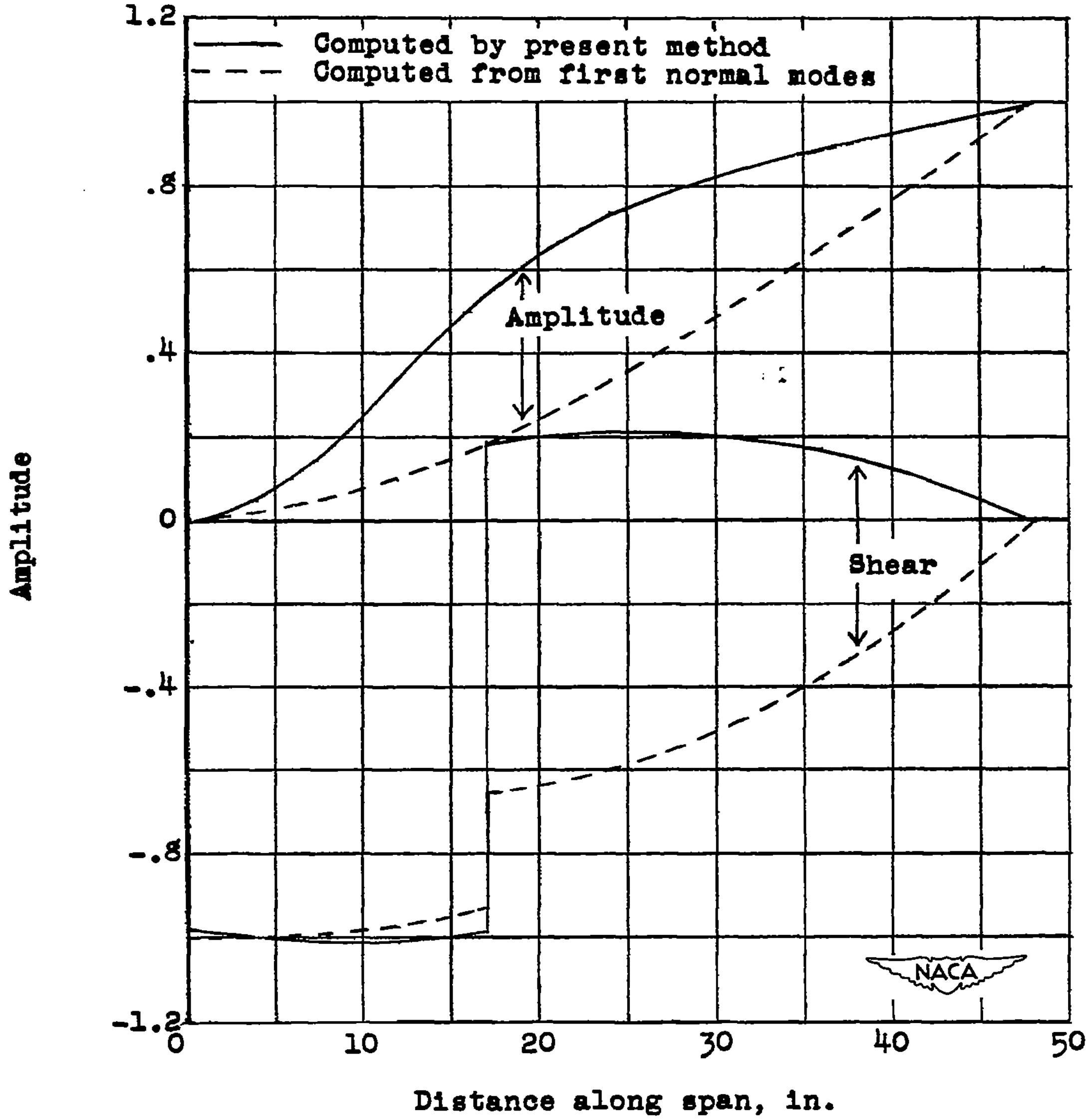


Figure 4.– Plots of amplitudes in bending displacement and torque and the corresponding curves computed for the first uncoupled normal mode in bending for $l_1 = 17$ inches (amplitude and shear referred to unit amplitude at the tip in bending).

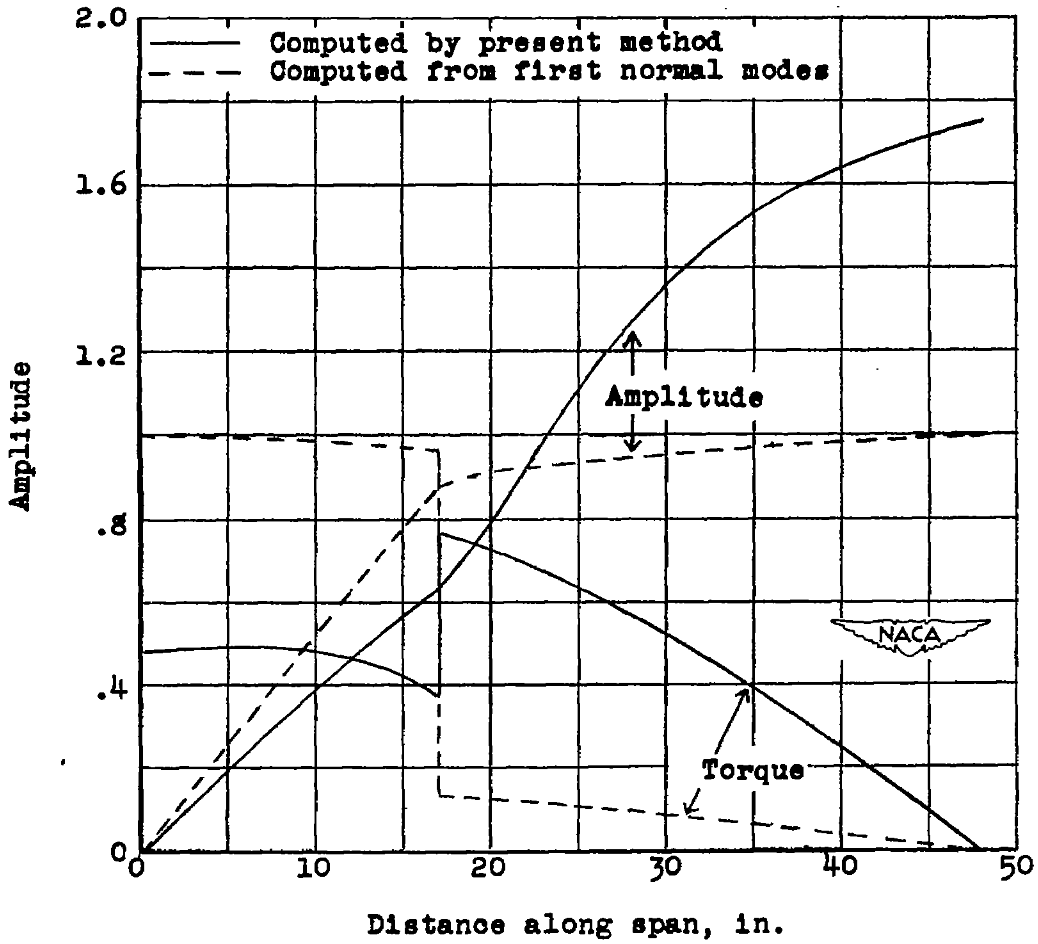


Figure 5.– Plots of amplitudes in torsional displacement and torque and the corresponding curves computed for the first uncoupled normal mode in torsion for $l_1 = 17$ inches (amplitude and torque referred to unit amplitude at tip in bending).

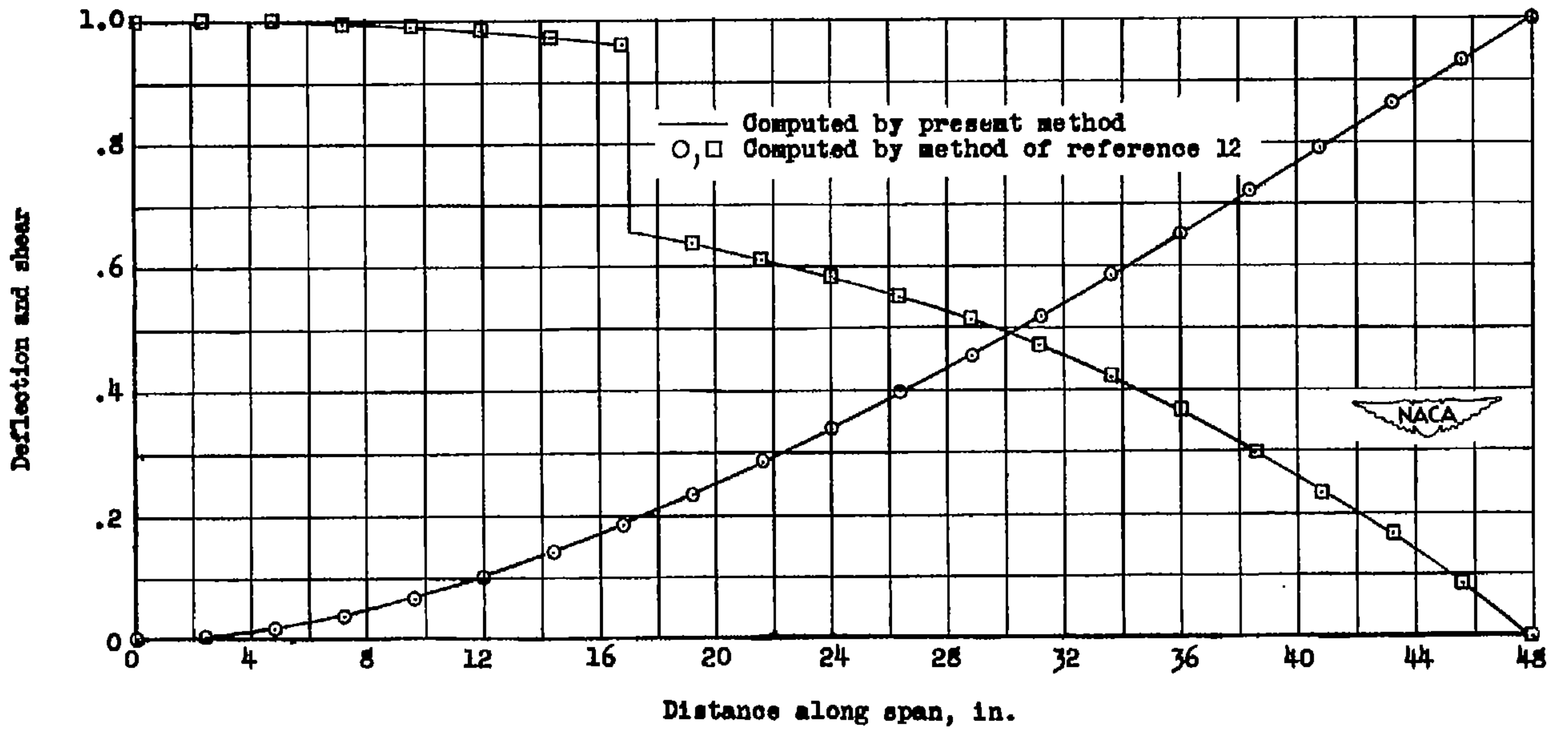
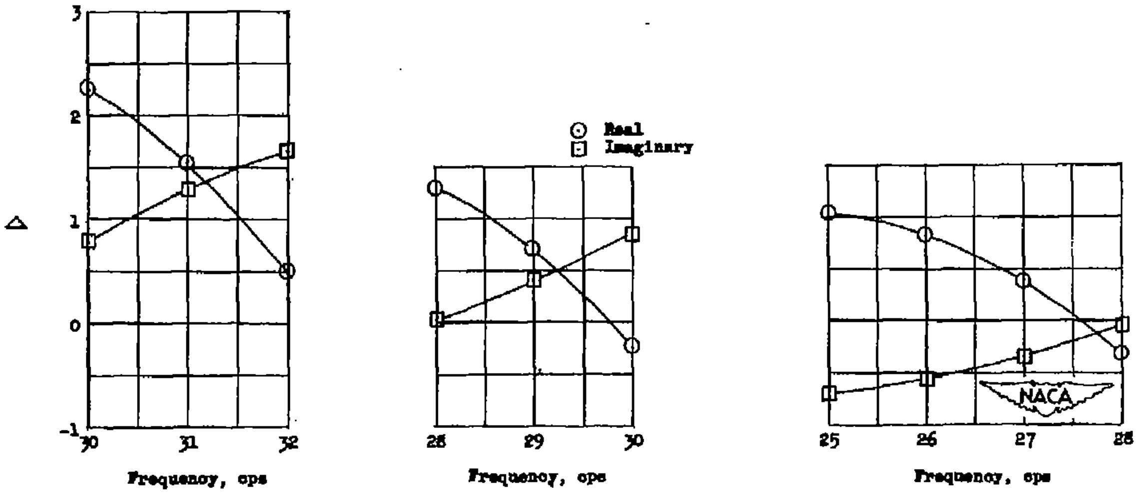
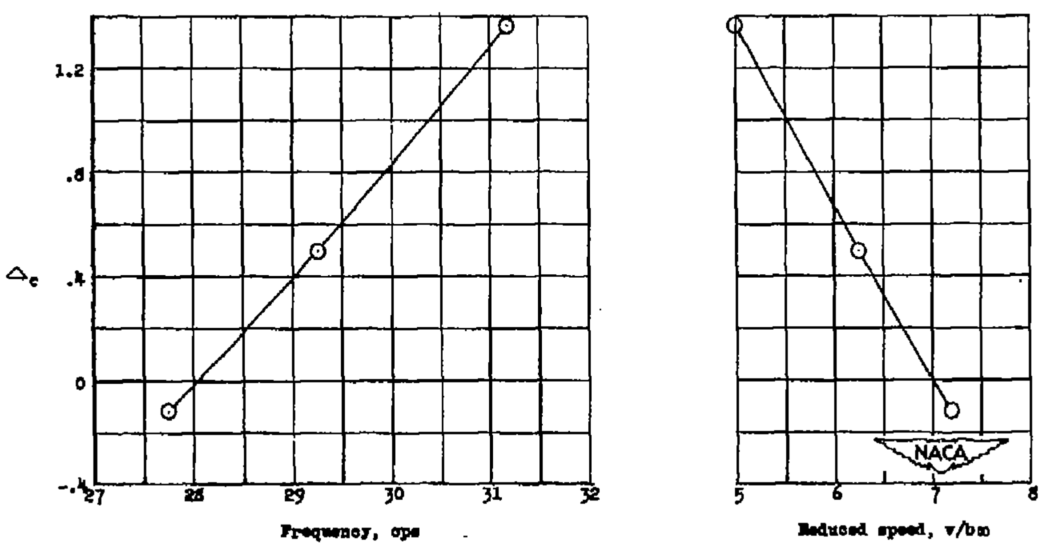


Figure 6.– Plots of deflection and shear curves computed from the first uncoupled modes in bending by the differential equation method and by the 20-station iteration process of reference 12 (referred to unit tip deflection).



(a) $\frac{v}{b\omega} = 5.00$. (b) $\frac{v}{b\omega} = 6.25$. (c) $\frac{v}{b\omega} = 7.1429$.

Figure 7.— Plots of Δ against frequency for particular values of reduced speed. ($l_1 = 17$ in.)



(a) Δ_e against frequency. (b) Δ_e against reduced speed.

Figure 8.— Plots of Δ_e against frequency and reduced speed. ($l_1 = 17$ in.)